

Solutions to *An Introduction to Manifolds*

September 13, 2026

Chapter 1

Euclidean Spaces

PROBLEM 1.2: A C^∞ function very flat at 0.

$$\text{Let } f(x) = \begin{cases} e^{-1/x} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

- (a) Show by induction that for $x > 0$ and $k \geq 0$, the k th derivative $f^{(k)}(x)$ is of the form $p_{2k}(1/x)e^{-1/x}$ for some polynomial $p_{2k}(y)$ of degree $2k$ in y .
- (b) Prove that f is C^∞ on $\mathbb{R}$ and that $f^{(k)}(0) = 0$ for all $k \geq 0$.

Solution:

- (a) The base case is trivial so assume $f^{(k)}(x) = p_{2k}\left(\frac{1}{x}\right)e^{-1/x}$. Then

$$\begin{aligned} f^{(k+1)} &= -\frac{1}{x^2} p'_{2k}\left(\frac{1}{x}\right) e^{-1/x} + p_{2k}\left(\frac{1}{x}\right) \cdot \left(-\frac{1}{x^2}\right) e^{-1/x} \\ &= \left(\frac{1}{x^2}\right) \left(-p'_{2k}\left(\frac{1}{x}\right) + p_{2k}\left(\frac{1}{x}\right)\right) \cdot e^{-1/x} \end{aligned}$$

Since $p'_{2k}(y)$ is of degree $2k - 1$ it cannot cancel the highest degree term in $p_{2k}(y)$ so that the sum is still of degree $2k$. The term $1/x^2$ corresponds to the polynomial y^2 of degree 2 so that multiplication with $-p'_{2k}(1/x) + p_{2k}(1/x)$ is a polynomial in $1/x$ of degree $2k + 2$.

- (b) Given a polynomial $p_n(y) = \sum_{i=0}^n c_i y^i$ we have

$$p_n(y) \leq cy^n$$

for $y \geq 1$ where $c = \max\{c_i\} \cdot (n + 1)$. Given a large enough y we can have that

$$cy^n \leq \varepsilon \frac{1}{(n + 1)!} y^{n+1}$$

for any ε . Specifically, choose any $y \geq \frac{c(n+1)!}{\varepsilon}$. Then

$$\varepsilon \frac{1}{(n+1)!} y^{n+1} < \varepsilon \sum_{n=0}^{\infty} \frac{1}{n!} y^n = \varepsilon e^y$$

also for $y \geq 1$. Combining these inequalities, we have that

$$p_n(y)e^{-y} < \varepsilon$$

as long as $y \geq \max\left\{1, \frac{c(n+1)!}{\varepsilon}\right\}$. Since ε may be arbitrarily small, we have that

$$\lim_{y \rightarrow \infty} p_n(y)e^{-y} = 0$$

By choosing $n = 2k$ and noting that

$$\lim_{x \rightarrow 0} \left(y = \frac{1}{x}\right) = \infty$$

we have that $f^{(k)}(0) = 0$ for all k .

PROBLEM 1.3: A diffeomorphism of an open interval with $\mathbb{R}$

Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^n$ be open subsets. A C^∞ map $F : U \rightarrow V$ is called a *diffeomorphism* if it is bijective and has a C^∞ inverse $F^{-1} : V \rightarrow U$.

- Show that the function $f :]-\pi/2, \pi/2[\rightarrow \mathbb{R}, f(x) = \tan x$ is a diffeomorphism.
- Let a, b be real numbers with $a < b$. Find a linear function $h :]a, b[\rightarrow]-1, 1[$, thus proving that any two finite open intervals are diffeomorphic.

SOLUTION:

- Given that

- $\lim_{x \rightarrow \frac{\pi}{2}} \cos x = 0$
- $0 < \sin c < \sin x$ for $0 < c < x < \frac{\pi}{2}$
- $0 < \cos x$ for $-\frac{\pi}{2} < x < \frac{\pi}{2}$

we have that $0 < \frac{\sin c}{\cos x} < \frac{\sin x}{\cos x}$ for $0 < c < x < \pi/2$, choosing some small non-zero c . Thus:

$$\infty = \lim_{x \rightarrow \frac{\pi}{2}} \frac{\sin c}{\cos x} < \lim_{x \rightarrow \frac{\pi}{2}} \frac{\sin x}{\cos x} = \lim_{x \rightarrow \frac{\pi}{2}} \tan x$$

Similarly, $\lim_{x \rightarrow -\frac{\pi}{2}} \tan x = -\infty$. Since $\tan x$ is differentiable, it is continuous so that it assumes all values in $\mathbb{R}$ by the intermediate value theorem.

The derivative of $\tan$ is

$$\begin{aligned}
 \tan' x &= \frac{\sin' x \cos x - \sin x \cos' x}{(\cos x)^2} \\
 &= \frac{(\cos x)^2 + (\sin x)^2}{(\cos x)^2} \\
 &= 1 + \left(\frac{\sin x}{\cos x} \right)^2 \\
 &= 1 + (\tan x)^2
 \end{aligned}$$

This is smooth since successive derivatives will be polynomials in $\tan x$. That is, assuming $\tan^{(k)} x = p_{k+1}(\tan x)$ where $p_{k+1}(x)$ is a polynomial of degree $k + 1$ in x , we have that

$$\begin{aligned}
 \tan^{(k+1)} x &= (1 + (\tan x)^2) p'_{k+1}(\tan x) \\
 &= p'_{k+1}(\tan x) + p'_{k+1}(\tan x) \cdot (\tan x)^2
 \end{aligned}$$

which is a polynomial in $\tan x$ of degree $k + 2$. Thus all the derivatives exist and are defined on $]-\pi/2, \pi/2[$.

Since $y^2 \geq 0$ for any $y \in \mathbb{R}$, we have that $1 + (\tan x)^2 \geq 1$. By the inverse function theorem, we have that the inverse $\tan^{-1} = \arctan$ exists for any neighborhood $U \subset \mathbb{R}$ of any point $y = \tan x$ for some $x \in]-\frac{\pi}{2}, \frac{\pi}{2}[$ which is all of $\mathbb{R}$ by the previous discussion. Since $x = \tan(\arctan x)$, we can calculate the derivative as follows:

$$\begin{aligned}
 1 &= \tan'(\arctan x) \cdot \arctan' x \\
 &= (1 + (\tan(\arctan x))^2) \cdot \arctan' x \\
 &= (1 + x^2) \cdot \arctan' x
 \end{aligned}$$

So that $\arctan' x = \frac{1}{1+x^2}$. By the fundamental theorem of calculus, we have that

$$\arctan x = \int_0^x \frac{1}{1+t^2} dt$$

given that $\arctan 0 = \arctan(\tan 0) = 0$. This is smooth because we can show that all the derivatives $\arctan^{(k)} x = \frac{p(x)}{(1+x^2)^k}$ where $p(x)$ is a

polynomial in x . Verifying this inductively:

$$\begin{aligned}\arctan^{(k+1)} &= \frac{p'(x) \cdot (1+x^2)^k - p(x) \cdot k(1+x^2)^{k-1} \cdot 2x}{(1+x^2)^{2k}} \\ &= \frac{(p'(x)(1+x^2) - 2kxp(x))(1+x^2)^{k-1}}{(1+x^2)^{2k}} \\ &= \frac{p'(x)(1+x^2) - 2kxp(x)}{(1+x^2)^{k+1}}\end{aligned}$$

Thus, since $1+x^2$ is never zero, all the derivatives $\arctan^{(k+1)}$ are defined and differentiable for all of $\mathbb{R}$. Thus $\tan$ is a diffeomorphism

$$\left]-\frac{\pi}{2}, -\frac{\pi}{2}\right[\xrightarrow{\cong} \mathbb{R}.$$

(b) Let $h(x) = \frac{2x-b-a}{b-a}$. Then

$$h(a) = \frac{2a-b-a}{b-a} = \frac{-(b-a)}{b-a} = -1$$

and

$$h(b) = \frac{2b-b-a}{b-a} = \frac{b-a}{b-a} = 1$$

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A DIGRESSION ON DERIVATIONS AND THE YONEDA LEMMA

A fundamental theorem of differential geometry is that the vector space of derivations at a point is homeomorphic $\mathbb{R}^n$. Somewhat surprisingly, this homeomorphism is a consequence of the Yoneda lemma, demonstrating the breadth of this lemma's application. In order to apply the Yoneda lemma all we must do is find the correct set-valued functor, which in this case is the derivative functor taking a smooth map to its Jacobian.

The crucial step of the proof is to show that a derivation at a point is a natural transformation which follows by applying the same fact, Taylor's theorem with remainder, which is used to establish the typical bijection between derivations and vectors.

Definition 1.1. Let $C_p^\infty(\mathbb{R}^n)$ be the set of germs of smooth functions on $\mathbb{R}^n$ at p with codomain $\mathbb{R}$. A **derivation at p** is a linear map $D : C_p^\infty(\mathbb{R}^n) \rightarrow \mathbb{R}$ such that

$$D(f \cdot g) = Df \cdot g(p) + f(p) \cdot Dg$$

Let C_*^∞ be the category with objects the euclidean spaces $\mathbb{R}^n$ with a chosen basepoint $p \in \mathbb{R}^n$, denoted $\mathbb{R}_p^n$, and with arrows $\mathbb{R}_p^n \rightarrow \mathbb{R}_q^m$ germs of smooth functions $f : U \rightarrow \mathbb{R}_q^m$ at $p \in U$ such that $q = f(p)$.

Theorem 1.2. *The set of derivations $\mathbf{D}(\mathbb{R}_p^n)$, define a functor $C_*^\infty \rightarrow \mathbf{Vect}$.*

Proof. Define addition and scalar multiplication on $\mathbf{D}(\mathbb{R}_p^n)$ pointwise. The proof that derivations form a vector space can be found in Loring Tu, *An Introduction to Manifolds* (technically an problem) so we only prove functoriality as follows.

Given $f \in C_*^\infty(\mathbb{R}_p^n, \mathbb{R}_q^m)$ define $\mathbf{D}(f) : \mathbf{D}(\mathbb{R}_p^n) \rightarrow \mathbf{D}(\mathbb{R}_q^m)$ by $f^*(D)(g) = D(gf)$ for $D \in \mathbf{D}(\mathbb{R}_p^n)$ and $g \in C_q^\infty(\mathbb{R}^m)$. This is a derivation $\mathbf{D}(\mathbb{R}_q^m)$ since

$$\begin{aligned} f^*(D)(g \cdot h) &= D((g \cdot h)f) \\ &= D(gf \cdot hf) \\ &= g(f(p)) \cdot D(hf) + D(gf) \cdot h(f(p)) \\ &= g(q) \cdot f^*(D)h + f^*(D)g \cdot h(q) \end{aligned}$$

□

Let $T_p(\mathbb{R}^n) \in \mathbf{Vect}$ be the tangent space at p . That is, $T_p(\mathbb{R}^n) = (\mathbb{R}^n, p)$, the vector space $\mathbb{R}^n$ coupled to a distinguished point $p \in \mathbb{R}^n$. Define the functor $T : C_*^\infty \rightarrow \mathbf{Vect}$ by $T(\mathbb{R}_p^n) = T_p\mathbb{R}^n$ on objects and on arrows by $T(f) = \Delta_p f$, the total derivative of f at p . Now, given any $v \in T_p\mathbb{R}^n$ s we can define a linear map $\delta_{\mathbb{R}_p^n}(v) : C_p^\infty(\mathbb{R}^n) \rightarrow \mathbb{R}$ by $\delta_{\mathbb{R}_p^n}(v)f = \Delta_p f \cdot v$. In fact we have that:

Theorem 1.3. *The map $\delta_{\mathbb{R}_p^n}(v) : C_p^\infty(\mathbb{R}^n) \rightarrow \mathbb{R}$ is a derivation and the functions $\delta_{\mathbb{R}_p^n} : T_p\mathbb{R}^n \rightarrow \mathbf{D}(\mathbb{R}_p^n)$ define a natural transformation $\delta : T \Rightarrow \mathbf{D}$.*

Proof. The map $\delta_{\mathbb{R}_p^n}(v)$ is a derivation at p because

$$\begin{aligned} \delta_{\mathbb{R}_p^n}(v)(f \cdot g) &= \Delta_p(f \cdot g) \cdot v \\ &= (g(p) \cdot \Delta_p f + f(p) \cdot \Delta_p g) \cdot v \\ &= g(p) \cdot \Delta_p f \cdot v + f(p) \cdot \Delta_p g \cdot v \\ &= g(p) \cdot \delta_{\mathbb{R}_p^n}(v)f + f(p) \cdot \delta_{\mathbb{R}_p^n}(v)g \\ &= \delta_{\mathbb{R}_p^n}(v)f \cdot g(p) + f(p) \cdot \delta_{\mathbb{R}_p^n}(v)g \end{aligned}$$

by the Leibniz product rule for total derivatives.

For naturality, we must show that, for $f \in C_*^\infty(\mathbb{R}_p^n, \mathbb{R}_q^m)$, the following diagram commutes

$$\begin{array}{ccc} T_p\mathbb{R}^n & \xrightarrow{\delta_{\mathbb{R}_p^n}} & \mathbf{D}(\mathbb{R}_p^n) \\ \Delta_p f \downarrow & & \downarrow \mathbf{D}(f) \\ T_q\mathbb{R}^m & \xrightarrow{\delta_{\mathbb{R}_q^m}} & \mathbf{D}(\mathbb{R}_q^m) \end{array}$$

Thus given any $v \in T_p \mathbb{R}^n$ and $g \in C_q^\infty(\mathbb{R}^m)$ we can compute, using the chain rule for total derivatives, that

$$\begin{aligned} \delta_{\mathbb{R}_q^m}(\Delta_p f \cdot v)g &= \Delta_q g \cdot \Delta_p f \cdot v \\ &= \Delta_p(gf) \cdot v \\ &= \delta_{\mathbb{R}_p^n}(v)(gf) \\ &= f^* \left(\delta_{\mathbb{R}_p^n}(v) \right) g \\ &= \mathbf{D}(f) \left(\delta_{\mathbb{R}_p^n}(v) \right) g \end{aligned}$$

so that $\delta_{\mathbb{R}_q^m} \circ \Delta_p f = \mathbf{D}(f) \circ \delta_{\mathbb{R}_p^n}$ □

Theorem 1.4. *Derivations at a point are naturally isomorphic to natural transformations $\mathbf{D} : C_*^\infty(\mathbb{R}_p^n, -) \Rightarrow T$.*

Proof. First we show that any natural transformation $\mathbf{D} : C_*^\infty(\mathbb{R}_p^n, -) \Rightarrow T$ defines a derivation D at p by $Df = \mathbf{D}_{\mathbb{R}_{f(p)}}(f)$ for $f \in C_p^\infty(\mathbb{R}^n)$. For the following proofs we will repeatedly refer to the following diagram

$$\begin{array}{ccc} C_*^\infty(\mathbb{R}_p^n, \mathbb{R}_p^n) & \xrightarrow{\mathbf{D}_{\mathbb{R}_p^n}} & T_p \mathbb{R}^n \\ f_* \downarrow & & \downarrow \Delta_p f \\ C_*^\infty(\mathbb{R}_p^n, \mathbb{R}_q) & \xrightarrow{\mathbf{D}_{\mathbb{R}_q}} & T_q \mathbb{R} \end{array} \quad (1.1)$$

and in particular the result of tracing the identity $1_{\mathbb{R}_p^n}(x) = x$ around the commutative square to give $\mathbf{D}_{\mathbb{R}_q} f = \Delta_p f \cdot \mathbf{D}_{\mathbb{R}_p^n} 1_{\mathbb{R}_p^n}$ for various values of $f : C_p^\infty(\mathbb{R}^n)$. First, taking $f(x) = g(x) + h(x)$ for two functions $g \in C_*^\infty(\mathbb{R}_p^n, \mathbb{R}_{g(p)})$ and $h \in C_*^\infty(\mathbb{R}_p^n, \mathbb{R}_{h(p)})$, we have that

$$\begin{aligned} \mathbf{D}_{\mathbb{R}_q}(g + h) &= \Delta_p(g + h) \cdot \mathbf{D}_{\mathbb{R}_p^n} 1_{\mathbb{R}_p^n} \\ &= (\Delta_p g + \Delta_p h) \cdot \mathbf{D}_{\mathbb{R}_p^n} 1_{\mathbb{R}_p^n} \\ &= \Delta_p g \cdot \mathbf{D}_{\mathbb{R}_p^n} 1_{\mathbb{R}_p^n} + \Delta_p h \cdot \mathbf{D}_{\mathbb{R}_p^n} 1_{\mathbb{R}_p^n} \\ &= \mathbf{D}_{\mathbb{R}_{g(p)}} g + \mathbf{D}_{\mathbb{R}_{h(p)}} h \end{aligned}$$

by linearity of the derivative. Thus D is linear.

Now, by taking $f(x) = g(x) \cdot f(x)$, we have

$$\begin{aligned} \mathbf{D}_{\mathbb{R}_q}(g \cdot h) &= \Delta_p(g \cdot h) \cdot \mathbf{D}_{\mathbb{R}_p^n} 1_{\mathbb{R}_p^n} \\ &= (g(p)\Delta_p h + h(p)\Delta_p g) \cdot \mathbf{D}_{\mathbb{R}_p^n} 1_{\mathbb{R}_p^n} \\ &= g(p)\Delta_p h \cdot \mathbf{D}_{\mathbb{R}_p^n} 1_{\mathbb{R}_p^n} + h(p)\Delta_p g \cdot \mathbf{D}_{\mathbb{R}_p^n} 1_{\mathbb{R}_p^n} \\ &= g(p)\mathbf{D}_{\mathbb{R}_{h(p)}} h + h(p)\mathbf{D}_{\mathbb{R}_{g(p)}} g \end{aligned}$$

by the Leibniz product rule for the derivative. Thus, $Df = \mathcal{D}_{\mathbb{R}_f(p)}(f)$ is a point-derivation at p .

Conversely, we show that a point-derivation D at p defines a natural transformation $\mathcal{D} : C_*^\infty(\mathbb{R}_p^n, -) \Rightarrow T$ as follows. Given $f \in C_*^\infty(\mathbb{R}_p^n, \mathbb{R}_q^m)$ we can write $f(x) = \sum f^j(x)e_j$ for m germs of functions $f^j \in C_p^\infty(\mathbb{R}^n)$, given a chosen basis e_j for $\mathbb{R}^n$. Define $\mathcal{D}_{\mathbb{R}_q^m} f = \sum Df^j e_j \in T_q \mathbb{R}^m$ which we will also write as Df or $D_x f(x)$. Note that this is linear and zero on constant functions since the same is true for D on each component (see Tu, *An Introduction to Manifolds*).

By Taylor's theorem with remainder, we can write f as

$$f(x) = f(p) + \sum (x^i - p^i) g_i(x)$$

where $x = \sum x^i e_i$, $p = \sum p^i e_i$, each $g_i \in C_*^\infty(\mathbb{R}_p^n, \mathbb{R}_{g_i(p)}^m)$, and $g_i(p) = \frac{\partial f}{\partial x^i}(p)$. Then we have that

$$\begin{aligned} \mathcal{D}_{\mathbb{R}_q^m} f &= D_x f(p) + \sum D_x ((x^i - p^i) g_i(x)) \\ &= 0 + \sum (p^i - p^i) \cdot D_x g_i(x) + D_x x^i \cdot g_i(p) \\ &= \sum D_x x^i \cdot \frac{\partial f}{\partial x^i}(p) \\ &= \Delta_p f \cdot D_x x \\ &= \Delta_p f \cdot \mathcal{D}_{\mathbb{R}_p^n} 1_{\mathbb{R}_p^n} \\ &= T(f) \cdot \mathcal{D}_{\mathbb{R}_p^n} 1_{\mathbb{R}_p^n} \end{aligned}$$

This shows that each $\mathcal{D}_{\mathbb{R}_q^m}$ is the same as the corresponding component of the natural transformation defined, due to the Yoneda lemma, by the element $\mathcal{D}_{\mathbb{R}_p^n} 1_{\mathbb{R}_p^n}$ so that $\mathcal{D}$ is natural.

Furthermore, this correspondence is a bijection since given $\mathcal{D} : C_*^\infty(\mathbb{R}_p^n, -) \Rightarrow T$ and defining $Df = \mathcal{D}_{\mathbb{R}_q^m} f$, we have that $Dx^i = \Delta_p x^i \mathcal{D}_{\mathbb{R}_p^n} 1_{\mathbb{R}_p^n}$ by taking $f(x) = x^i$ in diagram (1.1). Choosing α^j to be the dual basis to e_j we can write the linear transformation $\Delta_p x^i$ as $\sum_j \frac{\partial x^i}{\partial x^j} \alpha^j$. But $\frac{\partial x^i}{\partial x^j}$ is the Kronecker delta δ_j^i so that $\Delta_p x^i = \alpha^i$. Using this fact to recover the natural transformation, we have, by definition of D_x , that

$$\begin{aligned} D_x x &= \sum_i D_x x^i e_i \\ &= \sum_i (\Delta_p x^i \cdot \mathcal{D}_{\mathbb{R}_p^n} 1_{\mathbb{R}_p^n}) e_i \\ &= \sum_i \alpha^i (\mathcal{D}_{\mathbb{R}_p^n} 1_{\mathbb{R}_p^n}) e_i \\ &= \mathcal{D}_{\mathbb{R}_p^n} 1_{\mathbb{R}_p^n} \end{aligned}$$

Thus D defines the natural transformation determined by the same tangent vector $\mathbb{D}_{\mathbb{R}_p^n} 1_{\mathbb{R}_p^n}$ that, by Yoneda, must determine $\mathbb{D}$. The other direction is trivial since we can recover D as simply the one-dimensional component of the natural transformation $\mathbb{D}$ it defines. $\square$

By applying the Yoneda lemma to the previous theorem, we have:

Corollary 1.5. *The natural transformation $\delta : T \Rightarrow \mathbf{D}$ defined in Theorem 1.3 is an isomorphism.*

Proof. The Yoneda lemma gives us a natural isomorphism

$$\Psi : T_p \mathbb{R}^n \rightarrow \text{Hom}(C_*^\infty(\mathbb{R}_p^n, -), T)$$

defined by $\Psi(x)_{\mathbb{R}_q^m}(f) = \Delta_p f \cdot x$ for $x \in T_p \mathbb{R}^n$ and $f \in C_*^\infty(\mathbb{R}_p^n, \mathbb{R}_q^m)$. The previous theorem shows that the one-dimensional components of this transformation $\Psi(x)$ establish a bijection with derivations at p which are exactly the maps $\delta_{\mathbb{R}_p^n}$ from Theorem 1.3. $\square$

PROBLEM 2.1: Vector fields

Let X be the vector field $x\partial/\partial x + y\partial/\partial y$. and $f(x, y, z) = x^2 + y^2 + z^2$. Compute Xf .

$$Xf = x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = x \cdot 2x + y \cdot 2y = 2x^2 + 2y^2$$

PROBLEM 2.4: Product of derivations

Let A be an algebra over a field K . If D_1 and D_2 are derivations of A , show that $D_1 \cdot D_2$ is not necessarily a derivation (it is if D_1 or $D_2 = 0$), but $D_1 \cdot D_2 - D_2 \cdot D_1$ is always a derivation of A .

Take $f(x, y) = (x + 1)(y + 1) = xy - x + y + 1$ and let $D_1 = \frac{\partial}{\partial x} \Big|_0$ and $D_2 = \frac{\partial}{\partial y} \Big|_0$. Then, given that $\frac{\partial f}{\partial x} = y + 1$ and $\frac{\partial f}{\partial y} = x + 1$, we have

$$(D_1 \cdot D_2)f = \frac{\partial f}{\partial x} \Big|_0 \cdot \frac{\partial f}{\partial y} \Big|_0 = 1 \cdot 1 = 1$$

but this must equal

$$\begin{aligned} (D_1 \cdot D_2)f &= (x + 1) \cdot \frac{\partial(y + 1)}{\partial x} \Big|_0 \cdot \frac{\partial(y + 1)}{\partial y} \Big|_0 + \frac{\partial(x + 1)}{\partial x} \Big|_0 \cdot \frac{\partial(x + 1)}{\partial y} \Big|_0 \cdot (y + 1) \Big|_0 \\ &= 1 \cdot 0 \cdot 1 + 1 \cdot 0 \cdot 1 \\ &= 0 + 0 = 0 \end{aligned}$$

so that $D_1 \cdot D_2$ is not a derivation.

PROBLEM 3.1: Tensor product of covectors

Let $e_1, \dots, e_n$ be a basis for a vector space V and let $\alpha^1, \dots, \alpha^n$ be its dual basis in V^* . Suppose $[g_{ij}] \in \mathbb{R}^{n \times n}$ is an $n \times n$ matrix. Define a bilinear function $f : V \times V \rightarrow \mathbb{R}$ by

$$f(v, w) = \sum_{1 \leq i, j \leq n} g_{ij} v^i w^j$$

for $v = \sum v^i e_i$ and $w = \sum w^j e_j$ in V . Describe f in terms of the tensor products of α^i and α^j , $1 \leq i, j \leq n$.

SOLUTION:

$$f = \sum_{1 \leq i, j \leq n} g_{ij} \alpha^i \otimes \alpha^j$$

Further, if we define

$$h = \sum_{1 \leq i, j \leq n} g_{ij} \alpha^i \wedge \alpha^j$$

then we have that

$$\begin{aligned} h(v, w) &= \sum_{1 \leq i, j \leq n} g_{ij} (v^i w^j - w^i v^j) \\ &= \sum_{1 \leq i, j \leq n} g_{ij} v^i w^j - \sum_{1 \leq i, j \leq n} g_{ij} v^j w^i \\ &= \sum_{1 \leq i, j \leq n} g_{ij} v^i w^j - \sum_{1 \leq i, j \leq n} g_{ji} v^i w^j \\ &= \sum_{1 \leq i, j \leq n} (g_{ij} - g_{ji}) v^i w^j \end{aligned}$$

so that $h = \sum (g_{ij} - g_{ji}) \alpha^i \otimes \alpha^j$ is determined by the matrix $[h_{ij}] = [g_{ij} - g_{ji}]$ which has the properties $h_{ii} = 0$ and $h_{ij} = -h_{ji}$. Thus, such 2-covectors are determined by the upper-right values excluding the diagonal and there are $\binom{n}{2}$ choices for such values as expected.

PROBLEM 3.2: Hyperplanes

- Let V be a vector space of dimension n and $f : V \rightarrow \mathbb{R}$ a nonzero linear functional. Show that $\dim \ker f = n - 1$.
- Show that a nonzero linear functional on a vector space V is determined up to a multiplicative constant by its kernel, a hyperplane in V . In other words, if f and $g : V \rightarrow \mathbb{R}$ are nonzero linear functionals and $\ker f = \ker g$, then $g = cf$ for some constant $c \in \mathbb{R}$.

Solution:

- Let $e_1, \dots, e_n$ be a basis for V and $\alpha^1, \dots, \alpha^n$ the dual basis so that $f = \sum c_i \alpha^i$ for some $c_i \in \mathbb{R}$. Since f is not zero, not all of the c_i can be

zero so assume, without loss of generality, that $c_n \neq 0$. Then, given any $v = \sum v^i e_i \in \ker f$, we have $\sum c_i v^i = 0$ so that

$$v^n = - \sum_{i=1}^{n-1} \frac{c_i v^i}{c_n}$$

This shows that the $n-1$ vectors $f_i = e_i - \frac{c_i}{c_n} e_n$ for $1 \leq i \leq n-1$ span $\ker f$ since v is exactly the following sum

$$\begin{aligned} \sum_{i=1}^{n-1} v^i f_i &= \sum_{i=1}^{n-1} \left(v^i e_i - \frac{c_i v^i}{c_n} e_n \right) \\ &= \sum_{i=1}^{n-1} v^i e_i - \sum_{i=1}^{n-1} \frac{c_i v^i}{c_n} e_n \\ &= \sum_{i=1}^{n-1} v^i e_i + v^n e_n \\ &= v \end{aligned}$$

For linear independence, suppose that $w = \sum w^i f_i = 0$. Then

$$0 = \alpha^j(w) = \sum_{i=1}^{n-1} \left(w^i \alpha^j(e_i) - \frac{c_i w^i}{c_n} \alpha^j(e_n) \right) = \sum_{i=1}^{n-1} \left(w^i \delta_i^j - \frac{c_i w^i}{c_n} \delta_n^j \right) = w^j$$

for $1 \leq j \leq n-1$ so that the f_i are independent.

(b) For any $v \in V$ we have that

$$\begin{aligned} v &= \sum_{i=1}^n v^i e_i \\ &= \sum_{i=1}^{n-1} \left(v^i e_i - \frac{c_i v^i}{c_n} e_n \right) + \sum_{i=1}^{n-1} \frac{c_i v^i}{c_n} e_n + v^n e_n \\ &= \sum_{i=1}^{n-1} v^i f_i + \sum_{i=1}^{n-1} \frac{c_i v^i}{c_n} e_n + \frac{c_n v^n}{c_n} e_n \\ &= v_0 + \sum_{i=1}^n \frac{c_i v^i}{c_n} e_n \end{aligned}$$

where v_0 , being a linear combination of the basis vectors f_i , is in the kernel of f .

Now supposing $\ker g = \ker f$ and writing $g = \sum d_i \alpha^i$, we have that

$$g(v) = g(v_0) + \sum_{i=1}^n \frac{c_i v^i}{c_n} g(e_n) = \sum_{i=1}^n \frac{c_i v^i}{c_n} d_n = \sum_{i=1}^n \frac{d_n}{c_n} c_i v^i = \frac{d_n}{c_n} f(v)$$

PROBLEM 3.4: A characterization of alternating k -tensors

Let f be a k -tensor on a vector space V . Prove that f is alternating if and only if f changes sign whenever two successive arguments are interchanged.

$$f(\dots, v_{i+1}, v_i, \dots) = -f(\dots, v_i, v_{i+1}, \dots)$$

for $i = 1, \dots, k-1$.

SOLUTION: We show that any transposition $(n \ m)$ can be written as the product of an odd number of transpositions of the form $(i \ i+1)$. If $m > n+1$ then

$$(n \ m) = (n \ n+1)(n+1 \ m)(n \ n+1)$$

since this sends $n \mapsto n+1 \mapsto m$, $m \mapsto n+1 \mapsto n$, $n+1 \mapsto n \mapsto n+1$ and no other elements are mentioned. By induction on $m-n$ we have that $(n+1 \ m)$ is a product of an odd number of transpositions of the form $(i \ i+1)$ so that $(n \ m)$ is too. The base case $(n \ n+1)$ is trivial.

Then, writing an arbitrary permutation τ as a product of k transpositions $\prod_{i=1}^k (n_i \ m_i)$ we have that $\text{sgn } \tau = -1^k$. By writing each $(n_i \ m_i)$ as a product of $2k_i + 1$ transpositions of the form $(i \ i+1)$ we have that τ is a product of a total of

$$\sum_{i=1}^k (2k_i + 1) = 2 \left(\sum_{i=1}^k k_i \right) + k$$

such transpositions.

Now, assuming that each such transposition flips the sign of f and letting $j = \sum_i k_i$, we have that

$$\begin{aligned} \tau f &= -1^{2j+k} f \\ &= -1^k f \\ &= (\text{sgn } \tau) f \end{aligned}$$

so that f is alternating.

Conversely, if f is alternating then the sign is flipped by successive interchange since this is the transposition $(i \ i+1)$ and $\text{sgn}((i \ i+1)) = -1$.

PROBLEM 3.5: Another characterization of alternating k -tensors

Let f be a k -tensor on a vector space V . Prove that f is alternating if and only if $f(v_1, \dots, v_k) = 0$ whenever two of the vectors $v_1, \dots, v_k$ are equal.

If f is alternating and $v_i = v_j$ for two of the vectors then

$$\begin{aligned} f(v_1, \dots, v_i, \dots, v_j, \dots, v_k) &= -f(v_1, \dots, v_j, \dots, v_i, \dots, v_k) \\ &= -f(v_1, \dots, v_i, \dots, v_j, \dots, v_k) \end{aligned}$$

so that $2f(v_1, \dots, v_k) = 0$ and thus $f(v_1, \dots, v_k) = 0$ after dividing by 2.

Conversely, we have

$$\begin{aligned} f(v_1, \dots, v_i, \dots, v_j, \dots, v_k) + f(v_1, \dots, v_j, \dots, v_i, \dots, v_k) \\ = f(2v_1, \dots, v_i + v_j, \dots, v_j + v_i, \dots, 2v_k) \\ = 0 \end{aligned}$$

since $v_i + v_j = v_j + v_i$. Thus for any transposition τ , $\tau f(v_1, \dots, v_k) = -1 f(v_1, \dots, v_k)$ and writing an arbitrary permutation σ as a product of k transpositions we have $\sigma f = -1^k f = (\text{sgn } \sigma) f$ so that f is alternating.

This is sufficient to show that f is alternating if and only if $f(v_1, \dots, v_k) = 0$ whenever $v_1, \dots, v_k$ are linearly dependent. Thus assume that f is alternating and that, without loss of generality, that $v_k = \sum_{j < k} a^j v_j$. Then

$$\begin{aligned} f(v_1, \dots, v_k) &= f(v_1, \dots, \sum_{j < k} a^j v_j) \\ &= \sum_{j < k} a^j f(v_1, \dots, v_j, \dots, v_j) \\ &= 0 \end{aligned}$$

since every term in the sum is zero by the solution to this problem.

Conversely, use the same argument as above since equality (in this case of $v_i + v_j = v_j + v_i$) is a linear dependence.

PROBLEM 3.6: Wedge Product and Scalars

Let V be a vector space. For $a, b \in \mathbb{R}$, $f \in A_k(V)$, $g \in A_l(V)$, show that $af \wedge bg = (ab)f \wedge g$.

SOLUTION: By definition

$$\begin{aligned} af \wedge bg(v_1, \dots, v_{k+l}) &= \sum_{\sigma} (\text{sgn } \sigma) af(v_{\sigma(1)}, \dots, v_{\sigma(k)}) bg(v_{\sigma(k+1)}, \dots, v_{\sigma(k+l)}) \\ &= ab \sum_{\sigma} (\text{sgn } \sigma) f(v_{\sigma(1)}, \dots, v_{\sigma(k)}) g(v_{\sigma(k+1)}, \dots, v_{\sigma(k+l)}) \\ &= (ab)f \wedge g(v_1, \dots, v_{k+l}) \end{aligned}$$

where σ ranges over all (k, l) -shuffles.

PROBLEM 3.7: Transformation rule for a wedge product of covectors

Suppose two sets of covectors on a vector space V , $\beta^1, \dots, \beta^l$ and $\gamma^1, \dots, \gamma^l$, are related by

$$\beta^i = \sum_{j=1}^l \alpha_j^i \gamma^j, \quad i = 1, \dots, l$$

for a $k \times k$ matrix $A = [\alpha_j^i]$. Show that

$$\beta^1 \wedge \dots \wedge \beta^l = (\det A) \gamma^1 \wedge \dots \wedge \gamma^l$$

SOLUTION: By Proposition 3.27, given arbitrary vectors $v_1, \dots, v_l$ we have that

$$\bigwedge_{i=1}^l \beta^i(v_1, \dots, v_l) = \det [\beta^i(v_k)]$$

Since $\beta^i(v_k) = \sum_{j=1}^l \alpha_j^i \gamma^j(v_k)$ we have that

$$\begin{aligned} \bigwedge_{i=1}^l \beta^i(v_1, \dots, v_l) &= \det \left[\sum_{j=1}^l \alpha_j^i \gamma^j(v_k) \right] \\ &= \det ([\alpha_j^i] [\gamma^j(v_k)]) \\ &= \det A \det [\gamma^j(v_k)] \end{aligned}$$

and since $\det [\gamma^j(v_k)] = \bigwedge_{j=1}^l \gamma^j(v_1, \dots, v_l)$ again by Proposition 3.27, we have that

$$\beta^1 \wedge \dots \wedge \beta^l = (\det A) \gamma^1 \wedge \dots \wedge \gamma^l$$

since the chosen vectors were arbitrary.

PROBLEM 3.8: Transformation rule for k -covectors

Let f be a k -covector on a vector space V . Suppose two sets of vectors $u_1, \dots, u_k$ and $v_1, \dots, v_k$ in V are related by

$$u_j = \sum_{i=1}^k a_j^i v_i, \quad j = 1, \dots, k$$

for a $k \times k$ matrix $A = [a_j^i]$. Show that

$$f(u_1, \dots, u_k) = (\det A) f(v_1, \dots, v_k)$$

SOLUTION:

$$\begin{aligned} f(u_1, \dots, u_k) &= f \left(\sum_{i=1}^k a_1^i v_i, \dots, \sum_{i=1}^k a_k^i v_i \right) \\ &= \sum_{i_1=1}^k \dots \sum_{i_k=1}^k a_1^{i_1} \dots a_k^{i_k} f(v_{i_1}, \dots, v_{i_k}) \end{aligned}$$

Since f is alternating, any term in the above sum where $i_l = i_m$ (so that $v_{i_l} = v_{i_m}$) is zero by Exercise 3.5. Thus the only terms that remain are those for which all of the $i_1, \dots, i_k$ are distinct, which is to say a permutation σ of $1, \dots, k$. In that case, and because f is alternating, the above sum becomes

$$\begin{aligned} \sum_{\sigma \in S_k} a_1^{\sigma(1)} \dots a_k^{\sigma(k)} f(v_{\sigma(1)}, \dots, v_{\sigma(k)}) &= \sum_{\sigma \in S_k} (\text{sgn } \sigma) a_1^{\sigma(1)} \dots a_k^{\sigma(k)} f(v_1, \dots, v_k) \\ &= (\det A^\top) f(v_1, \dots, v_k) \\ &= (\det A) f(v_1, \dots, v_k) \end{aligned}$$

since the determinant of the transpose of a matrix is the same.

EXERCISE 4.9: (A closed 1-form on the punctured plane.)

Define a 1-form ω on $\mathbb{R}^2 - \{0\}$ by

$$\omega = \frac{1}{x^2 + y^2} (-ydx + xdy)$$

Show that ω is closed.

SOLUTION:

$$\begin{aligned} d\omega &= d\left(\frac{1}{x^2 + y^2} (-ydx + xdy)\right) \\ &= \frac{-(x^2 + y^2) - (-y)2y}{(x^2 + y^2)^2} dy \wedge dx + \frac{x^2 + y^2 - x2x}{(x^2 + y^2)^2} dx \wedge dy \\ &= \frac{-x^2 + y^2}{(x^2 + y^2)^2} dy \wedge dx + \frac{-x^2 + y^2}{(x^2 + y^2)^2} dx \wedge dy \\ &= \frac{x^2 - y^2 - x^2 + y^2}{(x^2 + y^2)^2} dx \wedge dy \\ &= 0 \end{aligned}$$

Chapter 2

Manifolds

PROBLEM 6.3: Group of automorphisms of a vector space.

Let V be a finite-dimensional vector space over $\mathbb{R}$, and $\text{GL}(V)$ the group of all linear automorphisms of V . Relative to an ordered basis $(e_1, \dots, e_n)$ for V , a linear automorphism for $L \in \text{GL}(V)$ is represented by a matrix $[a_j^i]$ defined by

$$Le_j = \sum_i a_j^i e_i$$

The map

$$\begin{aligned} \phi_e : \text{GL}(V) &\rightarrow \text{GL}(n, \mathbb{R}) \\ L &\mapsto [a_j^i] \end{aligned}$$

is a bijection with an open subset of $\mathbb{R}^{n \times n}$ that makes $\text{GL}(V)$ into a C^∞ manifold, which we denote temporarily by $\text{GL}(V)_e$. If $\text{GL}(V)_u$ is the manifold structure induced from another ordered basis $(u_1, \dots, u_n)$ for V , show that $\text{GL}(V)_e$ is the same as $\text{GL}(V)_u$.

SOLUTION: Note that if α^i is the dual basis to e_i then $\alpha^i Le_j = a_j^i$. Let β^i be the dual basis to u_i and let T be the linear automorphism defined by $Te_i = u_i$. Then we have that $T^{-1}u_i = e_i$ so that $\alpha^i T^{-1}u_j = \alpha^i e_j = \delta_j^i$. This demonstrates that $\beta^i = \alpha^i T^{-1}$. Let $[b_j^i]$ be the matrix representing L relative to u_i . Then $b_j^i = \beta^i Lu_j = \alpha^i T^{-1} L Te_j$. Let T and T^{-1} be represented by $[\tau_j^i]$ and $[\sigma_j^i]$ respectively, relative to e_i . By multiplying matrix entries, we then have that

$$b_j^i = \sum_{k,l} \sigma_k^i a_l^k \tau_j^l$$

showing that each b_j^i is a linear function of the coordinates a_j^i . This shows that these two charts are compatible and thus belong to the same maximal atlas.

Chapter 3

The Tangent Space

PROBLEM 8.1: Differential of a map

Let $F : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be the map

$$(u, v, w) = F(x, y) = (x, y, xy)$$

Let $p = (x, y) \in \mathbb{R}^2$. Compute $F_* \left(\frac{\partial}{\partial x} \Big|_p \right)$ as a linear combination of $\frac{\partial}{\partial u}$, $\frac{\partial}{\partial v}$, and $\frac{\partial}{\partial w}$ at $F(p)$.

SOLUTION: Following Proposition 8.11, F_* at p corresponds to the matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ y & x \end{bmatrix}$$

relative to the bases $\frac{\partial}{\partial x} \Big|_p$, $\frac{\partial}{\partial y} \Big|_p$ and $\frac{\partial}{\partial u} \Big|_{F(p)}$, $\frac{\partial}{\partial v} \Big|_{F(p)}$, $\frac{\partial}{\partial w} \Big|_{F(p)}$. Thus,

$$F_* \left(\frac{\partial}{\partial x} \Big|_p \right) = \frac{\partial}{\partial u} \Big|_{F(p)} + y \frac{\partial}{\partial w} \Big|_{F(p)}$$

and we can also easily compute

$$F_* \left(\frac{\partial}{\partial y} \Big|_p \right) = \frac{\partial}{\partial v} \Big|_{F(p)} + x \frac{\partial}{\partial w} \Big|_{F(p)}$$

PROBLEM 8.2: Differential of a linear map

Let $L : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear map. For any $p \in \mathbb{R}^n$, there is a canonical identification $T_p \mathbb{R}^n \rightarrow \mathbb{R}^n$ given by

$$\sum a^i \frac{\partial}{\partial x^i} \Big|_p \mapsto \mathbf{a} = \langle a^1, \dots, a^n \rangle$$

Show that the differential $L_* : T_p\mathbb{R}^n \rightarrow T_{L(p)}\mathbb{R}^m$ is the map $L : \mathbb{R}^n \rightarrow \mathbb{R}^m$ itself with the identification of the tangent spaces as above.

SOLUTION: We prove the following more general proposition:

Theorem 3.1. *Any linear map $L : T_p\mathbb{R}^n \rightarrow T_q\mathbb{R}^m$ corresponds, under the identification above, to a unique affine map $A : \mathbb{R}^n \rightarrow \mathbb{R}^m$ with $A(p) = q$ such that $A_* = L$. Furthermore, A is an isomorphism if and only if L is.*

This solves the problem since any linear map $L : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is affine so that L will equal the map A of the proposition.

Proof. Let $L : T_p\mathbb{R}^n \rightarrow T_q\mathbb{R}^m$ and let x^i be the standard coordinates on $\mathbb{R}^n$ and $\mathbb{R}^m$. Then we have that

$$L\left(\frac{\partial}{\partial x^j}\Big|_p\right) = \sum_{i=1}^m a_j^i \frac{\partial}{\partial x^i}\Big|_q$$

so that $L = [a_j^i]$ relative to the bases $\partial/\partial x^i|_p$ and $\partial/\partial x^i|_q$.

Given $p = \langle p^1, \dots, p^n \rangle$ and $q = \langle q^1, \dots, q^m \rangle$, let $o \in \mathbb{R}^m$ be defined by $o^i = q^i - \sum_j a_j^i p^j$ and define $A : \mathbb{R}^n \rightarrow \mathbb{R}^m$ by

$$A^i(x^1, \dots, x^n) = o^i + \sum_{j=1}^n a_j^i x^j$$

In other words, $A(x) = o + \bar{L}(x)$ where $\bar{L} : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is the linear transformation determined by the same matrix $[a_j^i]$ as L , but here relative to the standard bases of $\mathbb{R}^n$ and $\mathbb{R}^m$.

By Proposition 8.11 we have that

$$\begin{aligned} A_*\left(\frac{\partial}{\partial x^j}\Big|_p\right) &= \sum_{i=1}^m \frac{\partial A^i(x^1, \dots, x^n)}{\partial x^j} \frac{\partial}{\partial x^i}\Big|_q \\ &= \sum_{i=1}^m \frac{\partial (o^i + \sum_{k=1}^n a_k^i x^k)}{\partial x^j} \frac{\partial}{\partial x^i}\Big|_q \\ &= \sum_{i=1}^m \sum_{k=1}^n a_k^i \frac{\partial x^k}{\partial x^j} \frac{\partial}{\partial x^i}\Big|_q \\ &= \sum_{i=1}^m a_j^i \frac{\partial}{\partial x^i}\Big|_q \end{aligned}$$

so that $A_* = L$. Furthermore

$$A^i(p) = o^i + \sum_j a_j^i p^j = q^i - \sum_j a_j^i p^j + \sum_j a_j^i p^j = q^i$$

so that $A(p) = q$.

Finally if L is an isomorphism then $A^{-1}(y) = -\tilde{L}^{-1}(o) + \tilde{L}^{-1}(y)$ since

$$\begin{aligned} A^{-1}(A(x)) &= A^{-1}(o + \tilde{L}(x)) \\ &= -\tilde{L}^{-1}(o) + \tilde{L}^{-1}(o + \tilde{L}(x)) \\ &= -\tilde{L}^{-1}(o) + \tilde{L}^{-1}(o) + \tilde{L}^{-1}(\tilde{L}(x)) \\ &= x \end{aligned}$$

and

$$\begin{aligned} A(A^{-1}(y)) &= A(-\tilde{L}^{-1}(o) + \tilde{L}^{-1}(y)) \\ &= o + \tilde{L}(-\tilde{L}^{-1}(o) + \tilde{L}^{-1}(y)) \\ &= o - \tilde{L}(\tilde{L}^{-1}(o)) + \tilde{L}(\tilde{L}^{-1}(y)) \\ &= y \end{aligned}$$

I still have to prove the other direction, that A is unique and that L is an isomorphism if A is. $\square$

PROBLEM 8.3: Differential of a linear map

Fix a real number α and define $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by

$$\begin{bmatrix} u \\ v \end{bmatrix} = (u, v) = F(x, y) = \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Let $X = -y\partial/\partial x + x\partial/\partial y$ be a vector field on $\mathbb{R}^2$. If $p = (x, y) \in \mathbb{R}^2$ and $F_*(X_p) = (a\partial/\partial u + b\partial/\partial v)|_{F(p)}$, find a and b in terms of x, y and α .

SOLUTION: From Exercise 8.2, because F is linear, F_* at p is represented by the same matrix so that

$$\begin{bmatrix} a \\ b \end{bmatrix} = F_* \left(-y \frac{\partial}{\partial x} \Big|_p + x \frac{\partial}{\partial y} \Big|_p \right) = \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix} \begin{bmatrix} -y \\ x \end{bmatrix}$$

so that $a = -y \cos \alpha - x \sin \alpha$ and $b = -y \sin \alpha + x \cos \alpha$.

PROBLEM 8.4: Transition matrix for coordinate vectors

Let x, y be the standard coordinates on $\mathbb{R}^2$ and let U be the open set

$$U = \mathbb{R}^2 - \{(x, 0) \mid x \geq 0\}$$

On U the polar coordinates (r, θ) are uniquely defined by

$$\begin{aligned} x &= r \cos \theta \\ y &= r \sin \theta, \quad r > 0, \quad 0 < \theta < 2\pi \end{aligned}$$

Find $\partial/\partial r$ and $\partial/\partial\theta$ in terms of $\partial/\partial x$ and $\partial/\partial y$.

SOLUTION:

$$\begin{aligned}\frac{\partial}{\partial r} &= \frac{\partial x}{\partial r} \frac{\partial}{\partial x} + \frac{\partial y}{\partial r} \frac{\partial}{\partial y} \\ &= \frac{\partial(r \cos \theta)}{\partial r} \frac{\partial}{\partial x} + \frac{\partial(r \sin \theta)}{\partial r} \frac{\partial}{\partial y} \\ &= \cos \theta \frac{\partial}{\partial x} + \sin \theta \frac{\partial}{\partial y}\end{aligned}$$

and similarly

$$\begin{aligned}\frac{\partial}{\partial \theta} &= \frac{\partial x}{\partial \theta} \frac{\partial}{\partial x} + \frac{\partial y}{\partial \theta} \frac{\partial}{\partial y} \\ &= \frac{\partial(r \cos \theta)}{\partial \theta} \frac{\partial}{\partial x} + \frac{\partial(r \sin \theta)}{\partial \theta} \frac{\partial}{\partial y} \\ &= -r \sin \theta \frac{\partial}{\partial x} + r \cos \theta \frac{\partial}{\partial y}\end{aligned}$$

PROBLEM 8.5: Velocity of a curve in local coordinates

Prove Proposition 8.15.

SOLUTION: Given a chart (U, x^i) we have that

$$\begin{aligned}dx^i(c'(t_0)) &= c_* \left(\frac{d}{dt} \Big|_{t_0} \right) (x^i) \\ &= \frac{d}{dt} \Big|_{t_0} (x^i \circ c)(t) \\ &= \frac{d}{dt} \Big|_{t_0} c^i(t) \\ &= \dot{c}^i(t_0)\end{aligned}$$

so that $c'(t_0) = \sum \dot{c}^i(t_0) \frac{\partial}{\partial x^i}$.

PROBLEM 8.6: Velocity vector

Let $p = (x, y)$ be a point in $\mathbb{R}^2$. Then

$$c_p(t) = \begin{bmatrix} \cos 2t & -\sin 2t \\ \sin 2t & \cos 2t \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, \quad t \in \mathbb{R}$$

is a curve with initial point p in $\mathbb{R}^2$. Compute the velocity vector $c'_p(0)$.

SOLUTION: In the basis $\partial/\partial x, \partial/\partial y$ we have that

$$c'_p(t) = \begin{bmatrix} -2 \sin 2t & -2 \cos 2t \\ 2 \cos 2t & -2 \sin 2t \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, \quad t \in \mathbb{R}$$

so that

$$c'_p(0) = -2y \frac{\partial}{\partial x} + 2x \frac{\partial}{\partial y}$$

PROBLEM 8.7: Tangent space to a product

If M and N are manifolds, let $\pi_1 : M \times N \rightarrow M$ and $\pi_2 : M \times N \rightarrow N$ be the two projections. Prove that for $(p, q) \in M \times N$,

$$(\pi_{1*}, \pi_{2*}) : T_{(p,q)}(M \times N) \rightarrow T_p M \times T_q N$$

is an isomorphism.

SOLUTION: Let (U_p, ϕ) and (U_q, ψ) be charts around $p \in M$ and $q \in N$ respectively. Then by definition $(U_p \times U_q, (\phi, \psi))$ is a chart around $(p, q) \in M \times N$. Letting $x^i = r^i \circ \phi$, $y^i = r^i \circ \psi$, and $z^i = r^i \circ (\phi, \psi)$ be the coordinates on U_p and U_q and $U_p \times U_q$ we have that $\partial/\partial x^i$, $\partial/\partial y^i$ and $\partial/\partial z^i$ are bases for $T_p M$, $T_q N$ and $T_{(p,q)}(M \times N)$. Notice that

$$\begin{aligned} z^i &= x^i \circ \pi_1 = \pi_1^i & 1 \leq i \leq m \\ z^{m+i} &= y^i \circ \pi_2 = \pi_2^i & 1 \leq i \leq n \end{aligned}$$

so that by Proposition 8.11 we have

$$\begin{aligned} \pi_{1*} \left(\frac{\partial}{\partial z^j} \Big|_{(p,q)} \right) &= \sum_{i=1}^m \frac{\partial \pi_1^i}{\partial z^j}(p, q) \frac{\partial}{\partial x^i} \Big|_p \\ &= \sum_{i=1}^m \frac{\partial z^i}{\partial z^j}(p, q) \frac{\partial}{\partial x^i} \Big|_p \end{aligned}$$

and thus

$$\begin{aligned} \pi_{1*} \left(\frac{\partial}{\partial z^j} \Big|_{(p,q)} \right) &= \frac{\partial}{\partial x^j} \Big|_p & 1 \leq j \leq m \\ \pi_{1*} \left(\frac{\partial}{\partial z^{m+j}} \Big|_{(p,q)} \right) &= 0 & 1 \leq j \leq n \end{aligned}$$

Similarly we have

$$\begin{aligned} \pi_{2*} \left(\frac{\partial}{\partial z^j} \Big|_{(p,q)} \right) &= \sum_{i=1}^n \frac{\partial \pi_2^i}{\partial z^j}(p, q) \frac{\partial}{\partial y^i} \Big|_q \\ &= \sum_{i=1}^n \frac{\partial z^{m+i}}{\partial z^j}(p, q) \frac{\partial}{\partial y^i} \Big|_q \end{aligned}$$

so that

$$\begin{aligned}\pi_{2*}\left(\frac{\partial}{\partial z^j}\Big|_{(p,q)}\right) &= 0 & 1 \leq j \leq m \\ \pi_{2*}\left(\frac{\partial}{\partial z^{m+j}}\Big|_{(p,q)}\right) &= \frac{\partial}{\partial y^j}\Big|_q & 1 \leq j \leq n\end{aligned}$$

Thus

$$(\pi_{1*}, \pi_{2*})\left(\sum_{i=1}^{m+n} a^i \frac{\partial}{\partial z^i}\right) = \left(\sum_{i=1}^m a^i \frac{\partial}{\partial x^i}, \sum_{i=1}^n a^{m+i} \frac{\partial}{\partial y^i}\right)$$

which is a bijection since $(\partial/\partial x^i, 0), (0, \partial/\partial y^i)$ is a basis for $T_p M \times T_q N$.

Now we consider the image of a velocity vector under this isomorphism. Any curve $c :]a, b[\rightarrow M \times N$ starting at (p, q) can be written $c(t) = ((c_1(t), c_2(t)))$ where $c_1 = \pi_1 \circ c$ and $c_2 = \pi_2 \circ c$. Then c_1 and c_2 are curves in M and N starting at p and q respectively and by Proposition 8.18 we have that

$$\begin{aligned}(\pi_{1*}, \pi_{2*})(c'(0)) &= (\pi_{1*}(c'(0)), \pi_{2*}(c'(0))) \\ &= ((\pi_1 \circ c)'(0), (\pi_2 \circ c)'(0)) \\ &= (c'_1(0), c'_2(0))\end{aligned}$$

PROBLEM 8.8: Differentials of multiplication and inverse

Let G be a Lie group with multiplication map $\mu : G \times G \rightarrow G$, inverse map $\iota : G \rightarrow G$ and identity element e .

- (a) Show that the differential at the identity of the multiplication map μ is addition:

$$\begin{aligned}\mu_* : T_e G \times T_e G &\rightarrow T_e G \\ \mu_*(X_e, Y_e) &= X_e + Y_e\end{aligned}$$

- (b) Show that the differential at the identity of ι is the negative

$$\begin{aligned}\iota_* : T_e G &\rightarrow T_e G \\ \iota_*(X_e) &= -X_e\end{aligned}$$

SOLUTION:

- (a) To any curve $c :]a, b[\rightarrow G$ at e corresponds the curve $\bar{c}(t) = (c(t), e) \in G \times G$ at (e, e) . Given the previous problem we have that $\bar{c}'(0) = (c'(0), 0)$ since e is a constant. Since e is the identity, we have that $\mu(c(t), e) = c(t)$ which is to say that $\mu \circ \bar{c} = c$. Then we have by Proposition 8.18 that

$$\mu_*(c'(0), 0) = \mu_*(\bar{c}'(0)) = (\mu \circ \bar{c})'(0) = c'(0)$$

and similarly

$$\mu_*(\mathbf{0}, c'(0)) = c'(0)$$

Since μ_* is linear we have that

$$\begin{aligned}\mu_*(c'_1(0), c'_2(0)) &= \mu_*((c'_1(0), \mathbf{0}) + (\mathbf{0}, c'_2(0))) \\ &= \mu_*(c'_1(0), \mathbf{0}) + \mu_*(\mathbf{0}, c'_2(0)) \\ &= c'_1(0) + c'_2(0)\end{aligned}$$

Since a velocity vector $c'(0)$ can represent any vector $X_e \in T_e G$, we have that μ_* is addition.

- (b) To any curve $c(t) \in G$ at e , associate the curve $\bar{c}(t) = (c(t), \iota(c(t)))$ so that $\bar{c}'(0) = (c'(0), (\iota \circ c)'(0))$. Because ι is the inverse, we have that $\mu(\bar{c}(t)) = e$ is constant so that $(\mu \circ \bar{c})'(0) = \mathbf{0}$. Again by proposition 8.18, we have

$$\begin{aligned}\mu_*(c'(0), \iota_*(c'(0))) &= \mu_*(c'(0), (\iota \circ c)'(0)) \\ &= \mu_*(\bar{c}'(0)) \\ &= (\mu \circ \bar{c})'(0) \\ &= \mathbf{0}\end{aligned}$$

Since μ_* is addition and $c'(0)$ can represent any vector X_e we have that

$$X_e + \iota_*(X_e) = \mathbf{0}$$

so that $\iota_*(X_e) = -X_e$.

PROBLEM 8.9: Transforming vectors to coordinate vectors

Let $X_1, \dots, X_n$ be n vector fields on an open subset U of a manifold of dimension n . Suppose that at $p \in U$, the vectors $X_1(p), \dots, X_n(p)$ are linearly independent. Show that there is a chart $(V, x^1, \dots, x^n)$ about p such that $X_i(p) = \partial/\partial x^i|_p$ for $i = 1, \dots, n$.

SOLUTION: Let $(V, \psi) = (V, y^1, \dots, y^n)$ be a coordinate chart around p and let $L : T_p V \rightarrow T_p V$ be the linear transformation defined by

$$L\left(\frac{\partial}{\partial y^j}\bigg|_p\right) = X_j(p)$$

which is an isomorphism since the $X_j(p)$ form a basis. Then by Exercise 8.2, $\psi_* \circ L \circ \psi_*^{-1} : T_{\psi(p)} \mathbb{R}^n \rightarrow T_{\psi(p)} \mathbb{R}^n$ corresponds uniquely to an affine transformation $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$ with $A(\psi(p)) = \psi(p)$ such that $A_* = \psi_* \circ L \circ \psi_*^{-1}$. Equivalently,

$$L = \psi_*^{-1} \circ A_* \circ \psi_* = (\psi^{-1} \circ A \circ \psi)_* \quad (3.1)$$

Since L is an isomorphism, so is A so that $A^{-1} \circ \psi$ is a diffeomorphism. By Proposition 6.11, $(V, A^{-1} \circ \psi) = (V, x^1, \dots, x^n)$ is a chart in the differentiable structure of the manifold. By Proposition 8.8,

$$(A^{-1} \circ \psi)_* \left(\frac{\partial}{\partial x^j} \Big|_p \right) = \frac{\partial}{\partial r^j} \Big|_{\psi(p)}$$

so that by functoriality

$$\frac{\partial}{\partial x^j} \Big|_p = (A^{-1} \circ \psi)_*^{-1} \left(\frac{\partial}{\partial r^j} \Big|_{\psi(p)} \right) = (\psi^{-1} \circ A)_* \left(\frac{\partial}{\partial r^j} \Big|_{\psi(p)} \right) \quad (3.2)$$

Then, again by Proposition 8.8 and using (3.1) and (3.2), we now have that

$$\begin{aligned} X_j(p) &= L \left(\frac{\partial}{\partial y^j} \Big|_p \right) \\ &= (\psi^{-1} \circ A \circ \psi)_* \left(\frac{\partial}{\partial y^j} \Big|_p \right) \\ &= (\psi^{-1} \circ A)_* \left(\frac{\partial}{\partial r^j} \Big|_{\psi(p)} \right) \\ &= \frac{\partial}{\partial x^j} \Big|_p \end{aligned}$$

PROBLEM 9.1: Regular Values

Define $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ by

$$f(x, y) = x^3 - 6xy + y^2$$

Find all values $c \in \mathbb{R}$ for which the level set $f^{-1}(c)$ is a regular submanifold of $\mathbb{R}^2$.

SOLUTION: The Jacobian of f is

$$\begin{bmatrix} 3x^2 - 6y & 2y - 6x \end{bmatrix}$$

which has rank 0 when

$$\begin{aligned} 3x^2 - 6y &= 0 \\ 2y - 6x &= 0 \end{aligned}$$

which has solutions at

$$\begin{aligned} x &= 0, y = 0 \\ x &= 6, y = 18 \end{aligned}$$

These values belong to the level sets

$$0^3 - 6 \cdot 0 \cdot 0 + 0^2 = 0$$

$$6^3 - 6 \cdot 6 \cdot 18 + 18^2 = -108$$

In Figure 3.1, a red $\times$ marks the two critical points. Also shown are the level sets at $-300, -100, 0$ and 50 . Level sets greater than 0 , such as at 50 , are composed of one connected component. At 0 a loop forms around $(6, 18)$. At -108 this loop converges to a point and below -108 the loop component disappears and the sub-manifold is one connected component again such as the level set at -300 in the figure.

PROBLEM 9.3: Solution set of two equations

Is the solution set of the system of equations

$$x^3 + y^3 + z^3 = 1, \quad z = xy$$

in $\mathbb{R}^3$ a smooth manifold? Prove your answer.

SOLUTION: The solution set of these equations corresponds to the zero set of the map $\mathbb{R}^3 \rightarrow \mathbb{R}^2$ defined by $(x, y, z) \mapsto (x^3 + y^3 + z^3 - 1, z - xy)$. This map's Jacobian is

$$\begin{bmatrix} 3x^2 & 3y^2 & 3z^2 \\ -y & -x & 1 \end{bmatrix}$$

which has rank less than 2 when all 2×2 minors have determinant zero. This corresponds to the equations

$$-3x^3 + 3y^3 = 0 \tag{3.3}$$

$$3x^2 + 3yz^2 = 0 \tag{3.4}$$

$$3y^2 + 3xz^2 = 0 \tag{3.5}$$

Equation (3.3) gives that $x = y$ so that equations (3.4) and (3.5) both become

$$0 = 3x^2 + 3xz^2 = 3x(x + z^2)$$

and thus either $x = y = 0$ and z can be any value or $x = y = -z^2$. In other words, the potential critical points of this system of equations lie in the image of the two curves $\mathbb{R} \rightarrow \mathbb{R}^3$ defined by

$$z \mapsto (0, 0, z) \tag{3.6}$$

$$z \mapsto (-z^2, -z^2, z) \tag{3.7}$$

In case (3.6), z must also be 0 since $z = xy = 0 \cdot 0 = 0$ but $(0, 0, 0)$ does not satisfy $x^3 + y^3 + z^3 = 1$. In case (3.7), $z = xy = z^2 z^2 = z^4$ so that $z = 1$ and therefore $x = y = -1$. But the point $(-1, -1, 1)$ also does not satisfy $x^3 + y^3 + z^3 = 1$ so that no potential critical points satisfy the system of equations. Thus the solution set is a regular manifold by Theorem 9.9.

Figure 3.2 shows the two surfaces with their intersection drawn on each. Figure 3.3 shows only the curve

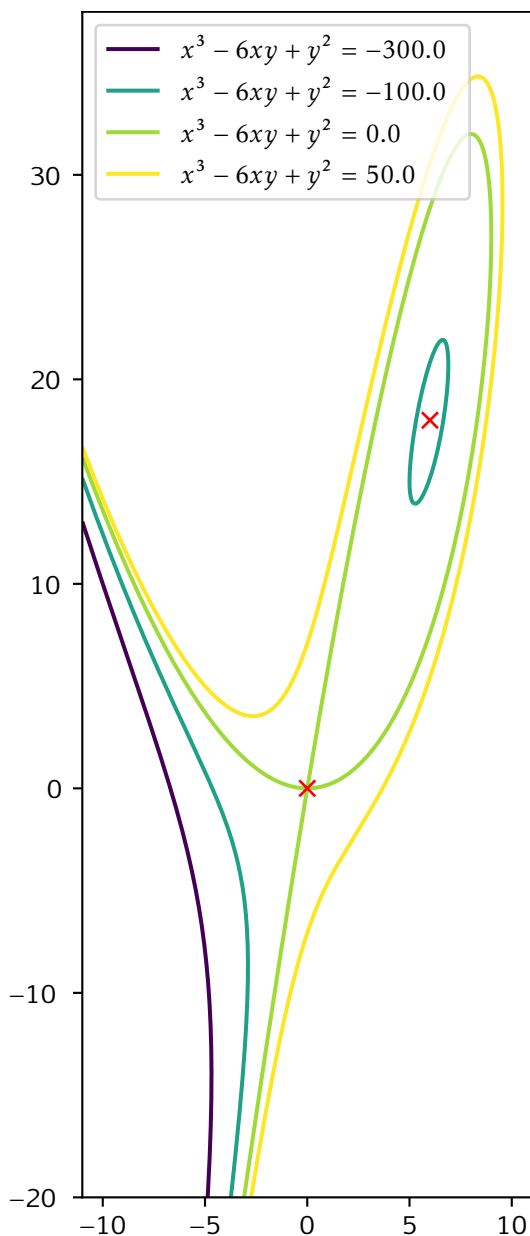


Figure 3.1: Level sets for $f(x, y) = x^3 - 6xy + y^2$

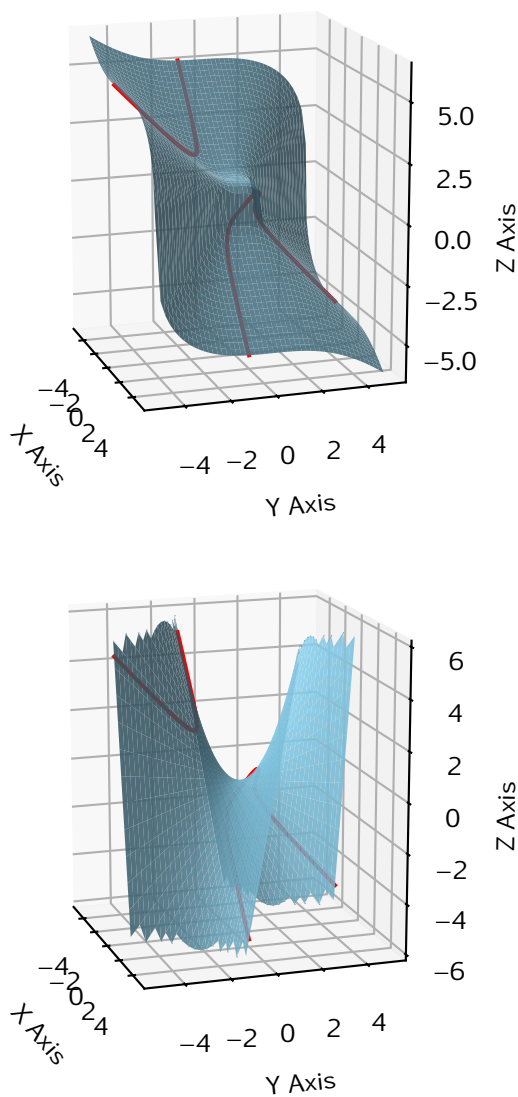


Figure 3.2: Surfaces $x^3 + y^3 + z^3 = 1$ and $z = xy$ with intersection curve

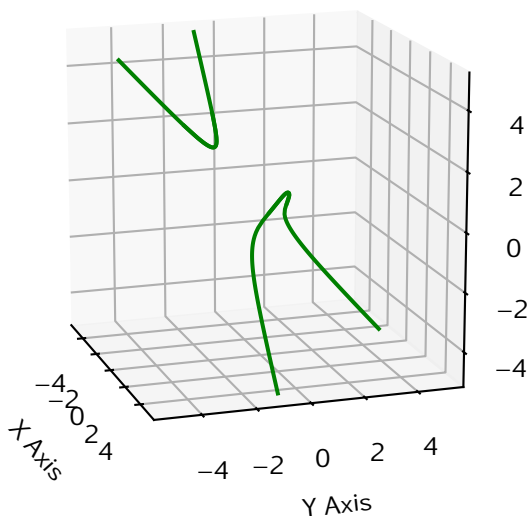


Figure 3.3: Intersection of $x^3 + y^3 + z^3 = 1$ and $z = xy$

PROBLEM 9.4: Regular submanifolds

Suppose that a subset S of $\mathbb{R}^2$ has the property that locally on S one of the coordinates is a C^∞ function of the other coordinate. Show that S is a regular submanifold of $\mathbb{R}^2$.

SOLUTION: Let $p \in S$ be such that there is a neighborhood $p \in U$ and a function $f : U_x \rightarrow U_y$ such that $S \cap U = \{(x, y) \mid y = f(x)\}$. Notice that the map $\mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by $(x, y) \mapsto (x, y - f(x))$ is a diffeomorphism since it has smooth inverse $(u, v) \mapsto (u, v + f(u))$. Since $S \cap U$ is the vanishing of $y - f(x)$, $(U, (x, y - f(x)))$ is an adapted chart around p . Since p was arbitrary S is a regular submanifold.

PROBLEM 9.5: Graph of a smooth function

Show that the graph $\Gamma(f)$ of a smooth function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$,

$$\Gamma(f) = \{(x, y, f(x, y)) \in \mathbb{R}^3\}$$

is a regular submanifold of $\mathbb{R}^3$.

SOLUTION: The function $\phi : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ defined by $\phi(x, y, z) = (x, y, z - f(x, y))$ is a diffeomorphism since it has smooth inverse $(u, v, w) \mapsto (u, v, w + f(u, v))$. Since $\Gamma(f)$ is defined by the vanishing of $z - f(x, y)$ this is an adapted chart relative to $\Gamma(f)$ so that it is a regular submanifold.

PROBLEM 9.6: Euler's Formula

A polynomial $F(x_0, \dots, x_n) \in \mathbb{R}[x_0, \dots, x_n]$ is *homogeneous of degree k* if it is a linear combination of monomials $x_0^{i_0} \cdots x_n^{i_n}$ of degree $\sum_{j=0}^n i_j = k$. Let $F(x_0, \dots, x_n)$ be a homogeneous polynomial of degree k . Clearly, for any $t \in \mathbb{R}$,

$$F(tx_0, \dots, tx_n) = t^k F(x_0, \dots, x_n) \quad (3.8)$$

Show that

$$\sum_{i=0}^n x_i \frac{\partial F}{\partial x_i} = kF \quad (3.9)$$

SOLUTION: Differentiating both sides of (3.8) gives that

$$\frac{d}{dt} F(tx_0, \dots, tx_n) = \sum_{i=0}^n x_i \frac{\partial F}{\partial x_i} (tx_0, \dots, tx_n)$$

and

$$\frac{d}{dt} t^k F(x_0, \dots, x_n) = kt^{k-1} F(x_0, \dots, x_n)$$

so that

$$\sum_{i=0}^n x_i \frac{\partial F}{\partial x_i} (tx_0, \dots, tx_n) = kt^{k-1} F(x_0, \dots, x_n)$$

Choosing $t = 1$ gives the result.

PROBLEM 9.7: Smooth projective hypersurface

Let $F(x_0, \dots, x_n)$ be a homogeneous polynomial of degree k . Show that the projective hypersurface $Z(F) \in \mathbb{RP}^n$ defined by $F(x_0, \dots, x_n) = 0$ is smooth if the $\partial F / \partial x_i$ are not all simultaneously zero on $Z(F)$.

SOLUTION: Let $[q] \in Z(F)$ and assume without loss of generality that

$$[q] \in U_0 = \{[x_0, \dots, x_n] \mid x_0 \neq 0\}$$

so we can write $[q] = [1, p_1, \dots, p_n] = [\iota_0(p)]$ where $p = (p_1, \dots, p_n) \in \mathbb{R}^n$ and $\iota_0(p_1, \dots, p_n) = (1, p_1, \dots, p_n)$.

Define $f_0 = F \circ \iota_0 : \mathbb{R}^n \rightarrow \mathbb{R}$. Then on the chart U_0 , ϕ_0 with $\phi_0(x_0, \dots, x_n) = (1, x_1/x_0, \dots, x_n/x_0)$, $Z(F) \cap U_0$ is the zero set of f_0 since

$$F(x_0, \dots, x_n) = x_0^k \cdot F(1, x_1/x_0, \dots, x_n/x_0) = x_0^k \cdot f_0(x_1/x_0, \dots, x_n/x_0)$$

implies that $f_0(x_1/x_0, \dots, x_n/x_0) = 0$ if and only if $F(x_0, \dots, x_n) = 0$ when-

ever $x_0 \neq 0$. Also notice that by the chain rule we have that

$$\begin{aligned}\frac{\partial f_0}{\partial x_i}(p) &= \sum_j \frac{\partial F}{\partial x_j}(\iota_0(p)) \frac{\partial \iota_0^j}{\partial x_i}(p) \\ &= \sum_j \frac{\partial F}{\partial x_j}(\iota_0(p)) \frac{\partial x_j}{\partial x_i}(p) \\ &= \frac{\partial F}{\partial x_i}(\iota_0(p))\end{aligned}$$

for $i = 1, \dots, n$.

Using equation (3.9) with $x_0 = 1$ we have that

$$\frac{\partial F}{\partial x_0}(\iota_0(p)) + \sum_{i=1}^n p_i \frac{\partial F}{\partial x_i}(\iota_0(p)) = k f_0(p) = 0$$

since $p \in Z(f_0)$. Thus

$$\begin{aligned}\frac{\partial F}{\partial x_0}(\iota_0(p)) &= - \sum_{i=1}^n p_i \frac{\partial F}{\partial x_i}(\iota_0(p)) \\ &= - \sum_{i=1}^n p_i \frac{\partial f_0}{\partial x_i}(p)\end{aligned}$$

Thus not all of the $\frac{\partial f_0}{\partial x_i}(p)$ can be zero since that would imply that each

$$\frac{\partial F}{\partial x_i}(\iota_0(p)) = \frac{\partial f_0}{\partial x_i}(p) = 0$$

and also that

$$\frac{\partial F}{\partial x_0}(\iota_0(p)) = - \sum p_i \frac{\partial f_0}{\partial x_i}(p) = - \sum p_i \cdot 0 = 0$$

so that all the $\partial F / \partial x_i$ are zero at $p \in Z(F)$ which we assume not to be the case. Thus by Corollary 9.10, there is an adapted chart $V \subset U_0$ for $Z(F)$ around p . Since p was arbitrary, $Z(F)$ is a regular submanifold of $\mathbb{R}P^n$.

PROBLEM 11.1: Tangent vectors to a sphere

The unit sphere S^n in $\mathbb{R}^{n+1}$ is defined by the equation $\sum_{i=1}^{n+1} (x^i)^2 = 1$. For $p = (p^1, \dots, p^{n+1}) \in S^n$, show that a necessary and sufficient condition for

$$X_p = \sum a^i \frac{\partial}{\partial x^i} \Big|_p \in T_p(\mathbb{R}^{n+1})$$

to be tangent to S^n at p is $\sum a^i p^i = 0$.

SOLUTION: Let $c(t) : \mathbb{R} \rightarrow \mathbb{R}^{n+1}$ lie in S^n such that $c(0) = p$ and $c'(0) = X_p$. Since $c(t) \in S^n$ we have that $\sum c^i(t)^2 = 1$ and thus by the chain rule

$$\sum_{i=1}^{n+1} 2c^i(t)\dot{c}^i(t) = 0$$

Dividing by 2 and taking $t = 0$ gives $\sum p^i a^i = 0$ so that the condition is necessary. By Exercise 3.2 we know that the hyperplane $H_p = \{\sum a^i e_i \mid \sum a^i p^i = 0\}$ has dimension n . Thus since $i_*(T_p S^n) \subset H_p$ also has dimension n , they must be equal so that the condition is sufficient.

PROBLEM 11.2: Tangent vectors to a plane curve

- (a) Let $i : S^1 \hookrightarrow \mathbb{R}^2$ be the inclusion map of the unit circle. In this problem, we denote by x, y the standard coordinates on $\mathbb{R}^2$ and by $\bar{x}, \bar{y}$ their restrictions to S^1 . Thus, $\bar{x} = i^*x = x \circ i$ and $\bar{y} = i^*y = y \circ i$. On the upper semicircle $U = \{(a, b) \in S^1 \mid b > 0\}$, $\bar{x}$ is a local coordinate so that $\partial/\partial\bar{x}$ is defined. Prove that for $p \in U$,

$$i_* \left(\frac{\partial}{\partial\bar{x}} \Big|_p \right) = \left(\frac{\partial}{\partial x} + \frac{\partial\bar{y}}{\partial\bar{x}} \frac{\partial}{\partial y} \right) \Big|_p$$

Thus, although $i_* : T_p S^1 \rightarrow T_p \mathbb{R}^2$ is injective, $\partial/\partial\bar{x}|_p$ cannot be identified with $\partial/\partial x|_p$.

- (b) Generalize (i) to a smooth curve C in $\mathbb{R}^2$, letting U be a chart in C on which $\bar{x}$, the restriction of x to C is a local coordinate.

SOLUTION:

- (a) By Proposition 8.11 we have that

$$\begin{aligned} i_* \left(\frac{\partial}{\partial\bar{x}} \Big|_p \right) &= \left(\frac{\partial x \circ i}{\partial\bar{x}}(p) \frac{\partial}{\partial x} + \frac{\partial y \circ i}{\partial\bar{x}}(p) \frac{\partial}{\partial y} \right) \Big|_p \\ &= \left(\frac{\partial\bar{x}}{\partial\bar{x}}(p) \frac{\partial}{\partial x} + \frac{\partial\bar{y}}{\partial\bar{x}}(p) \frac{\partial}{\partial y} \right) \Big|_p \\ &= \left(\frac{\partial}{\partial x} + \frac{\partial\bar{y}}{\partial\bar{x}}(p) \frac{\partial}{\partial y} \right) \Big|_p \end{aligned}$$

In this case since $\bar{y} = \sqrt{1 - \bar{x}^2}$, and letting $p = (a, b)$ we have that

$$\begin{aligned} i_* \left(\frac{\partial}{\partial\bar{x}} \Big|_p \right) &= \left(\frac{\partial}{\partial x} - \frac{a}{\sqrt{1-a^2}} \frac{\partial}{\partial y} \right) \Big|_p \\ &= \left(\frac{\partial}{\partial x} - \frac{a}{b} \frac{\partial}{\partial y} \right) \Big|_p \end{aligned}$$

(b) Assume $C(t_0) = p \in U$ and let $\bar{x} = x \circ C, \bar{y} = y \circ C$ so that

$$C(t) = (\bar{x}(t), \bar{y}(t)) \quad (3.10)$$

and

$$C'(t_0) = \left(a \frac{\partial}{\partial x} + b \frac{\partial}{\partial y} \right) \Big|_p$$

where $a = \frac{d\bar{x}}{dt}(t_0)$ and $b = \frac{d\bar{y}}{dt}(t_0)$. Since $\bar{x}$ is a local coordinate, $\bar{x}(t) : \mathbb{R} \rightarrow \mathbb{R}$ is a local diffeomorphism in U so that, firstly a is non-zero and secondly there exists a local inverse $\bar{x}^{-1}$ such that $\frac{d\bar{x}^{-1}}{dx}(x_0) = \frac{1}{a}$ where $\bar{x}(t_0) = x_0$. Then by (3.10)

$$\begin{aligned} (C \circ \bar{x}^{-1})(x) &= C(\bar{x}^{-1}(x)) \\ &= (\bar{x}(\bar{x}^{-1}(x)), \bar{y}(\bar{x}^{-1}(x))) \\ &= (x, (\bar{y} \circ \bar{x}^{-1})(x)) \end{aligned}$$

Then we can say the following: $(x, y - \bar{y}(\bar{x}^{-1}(x)))$ is an adapted chart for the image of C in $\mathbb{R}^2$ and $C \circ \bar{x}^{-1}$ is the inclusion map relative to the local coordinate $\bar{x}$ of this adapted chart. Thus we have resolved $\bar{x}$ and $\bar{y}$ to be functions of $\bar{x}$ instead of t so that by the chain rule we can now compute

$$\begin{aligned} i_* \left(\frac{\partial}{\partial \bar{x}} \Big|_p \right) &= \left(\frac{\partial}{\partial x} + \frac{\partial \bar{y}}{\partial \bar{x}}(x_0) \frac{\partial}{\partial y} \right) \Big|_p \\ &= \left(\frac{\partial}{\partial x} + \frac{d\bar{y} \circ \bar{x}^{-1}}{dx}(x_0) \frac{\partial}{\partial y} \right) \Big|_p \\ &= \left(\frac{\partial}{\partial x} + \frac{d\bar{y}}{dt}(t_0) \cdot \frac{d\bar{x}^{-1}}{dx}(x_0) \frac{\partial}{\partial y} \right) \Big|_p \\ &= \left(\frac{\partial}{\partial x} + \frac{b}{a} \frac{\partial}{\partial y} \right) \Big|_p \end{aligned}$$

so that

$$i_* \left(\frac{\partial}{\partial \bar{x}} \Big|_p \right) = \frac{1}{a} C'(t_0)$$

thus normalizing the vector $C'(t_0)$ on its first component.

PROBLEM 11.3: Critical points of a smooth map on a compact manifold

Show that a smooth map f from a compact manifold N to $\mathbb{R}^m$ has a critical point.

SOLUTION: By Proposition A.34, the image of f is compact and therefore closed and bounded by the Heine-Borel Theorem, Theorem A.40. Thus the

image of f is a proper subset of $\mathbb{R}^m$. By Proposition A.35, f is a closed map. Now suppose f has no critical points so that f is everywhere a submersion. By Corollary 11.6, f is an open map. Since N is both open and closed, the proper subset $f(N)$ is also open and closed. In particular, $\mathbb{R}^m \setminus f(N)$ is also open so that $\mathbb{R}^m$ is the disjoint union of two nonempty subsets and therefore disconnected. But $\mathbb{R}^m$ is connected so this is a contradiction.

PROBLEM 11.4: Differential of an inclusion map

On the upper hemisphere of the unit sphere S^2 , we have the coordinate map $\phi = (u, v)$, where

$$u(a, b, c) = a \quad \text{and} \quad v(a, b, c) = b$$

So the derivations $\partial/\partial u|_p, \partial/\partial v|_p$ are tangent vectors of S^2 at any point $p = (a, b, c)$ on the upper hemisphere. Let $i : S^2 \rightarrow \mathbb{R}^3$ be the inclusion and x, y, z the standard coordinates on $\mathbb{R}^3$. The differential $i_* : T_p S^2 \rightarrow T_p \mathbb{R}^3$ maps $\partial/\partial u|_p, \partial/\partial v|_p$ into $T_p \mathbb{R}^3$. Thus,

$$\begin{aligned} i_* \left(\frac{\partial}{\partial u} \Big|_p \right) &= \alpha^1 \frac{\partial}{\partial x} \Big|_p + \beta^1 \frac{\partial}{\partial y} \Big|_p + \gamma^1 \frac{\partial}{\partial z} \Big|_p \\ i_* \left(\frac{\partial}{\partial v} \Big|_p \right) &= \alpha^2 \frac{\partial}{\partial x} \Big|_p + \beta^2 \frac{\partial}{\partial y} \Big|_p + \gamma^2 \frac{\partial}{\partial z} \Big|_p \end{aligned}$$

for some constants $\alpha^i, \beta^i, \gamma^i$. Find $(\alpha^i, \beta^i, \gamma^i)$, for $i = 1, 2$.

SOLUTION: The inverse map $\phi^{-1} = (x, y, z)$ is

$$x(a, b) = a, \quad y(a, b) = b, \quad z(a, b) = \sqrt{1 - a^2 - b^2} = c$$

So that by Proposition 8.11,

$$\begin{aligned} \alpha^1 &= \frac{\partial x}{\partial u}(a, b) = \frac{\partial u}{\partial u}(a, b) = 1 \\ \alpha^2 &= \frac{\partial y}{\partial u}(a, b) = \frac{\partial v}{\partial u}(a, b) = 0 \\ \alpha^3 &= \frac{\partial z}{\partial u}(a, b) = \frac{\partial \sqrt{1 - u^2 - v^2}}{\partial u}(a, b) = -\frac{a}{\sqrt{1 - a^2 - b^2}} = -\frac{a}{c} \end{aligned}$$

and similarly

$$\begin{aligned} \beta^1 &= \frac{\partial x}{\partial v}(p) = 0 \\ \beta^2 &= \frac{\partial y}{\partial v}(p) = 1 \\ \beta^3 &= \frac{\partial z}{\partial v}(p) = -\frac{b}{\sqrt{1 - a^2 - b^2}} = -\frac{b}{c} \end{aligned}$$

PROBLEM 11.5: One-to-one immersion of a compact manifold

Prove that if N is a compact manifold, then a one-to-one immersion $f : N \rightarrow M$ is an embedding.

SOLUTION: First note that f restricts to a surjective continuous map onto its image $f : N \rightarrow f(N)$ where $f(N)$ is given the relative topology since if $U \cap f(N)$ is an open subset of $f(N)$ for some open set $U \subset M$, then $f^{-1}(U)$ is open in N because f is continuous and $f^{-1}(U) = f^{-1}(U \cap f(N))$. Because f is one-to-one, it is a continuous bijection to its image. Because N is compact, f is a homeomorphism by Corollary A.35 thus satisfying the second requirement to be an embedding.

PROBLEM 12.1: Hausdorff condition on the tangent bundle

Prove that the tangent bundle TM of a manifold M is Hausdorff.

SOLUTION: Let $\pi : TM \rightarrow M$ be the tangent bundle and let $x_1 \neq x_2 \in TM$. We consider two cases

- (a) $\pi(x_1) = \pi(x_2)$
- (b) $\pi(x_1) \neq \pi(x_2)$

In the first case there is a trivializing open set U containing $\pi(x_1) = \pi(x_2) = q$ so that $\phi : E|_U \rightarrow U \times \mathbb{R}^m$ is a diffeomorphism. Then we have that $\phi(x_1) = (q, \sum c^i(x_1)e_i)$ and $\phi(x_2) = (q, \sum c^i(x_2)e_i)$. Since ϕ is a diffeomorphism, $\phi(x_1) \neq \phi(x_2)$ so that $\sum c^i(x_1)e_i \neq \sum c^i(x_2)e_i$. Because $\mathbb{R}^m$ is Hausdorff, there are disjoint open sets U_1 and U_2 in $\mathbb{R}^m$ containing $\sum c^i(x_1)e_i$ and $\sum c^i(x_2)e_i$ respectively. Then $\phi^{-1}(U \times U_1)$ and $\phi^{-1}(U \times U_2)$ are disjoint open sets containing x_1 and x_2 respectively.

In the second case, there are disjoint open sets U_1 and U_2 in M that contain $\pi(x_1)$ and $\pi(x_2)$ respectively in which case $\pi^{-1}(U_1)$ and $\pi^{-1}(U_2)$ are disjoint open sets that contain x_1 and x_2 .

PROBLEM 12.2: Let $(U, \phi) = (U, x^1, \dots, x^n)$

and $(V, \psi) = (V, y^1, \dots, y^n)$ be overlapping coordinate charts on a manifold M . They induce coordinate charts $(TU, \tilde{\phi})$ and $(TV, \tilde{\psi})$ on the total space TM of the tangent bundle with transition function $\tilde{\psi} \circ \tilde{\phi}^{-1}$:

$$(x^1, \dots, x^n, a^1, \dots, a^n) \mapsto (y^1, \dots, y^n, b^1, \dots, b^n)$$

- (a) Compute the Jacobian matrix of the transition function $\tilde{\psi} \circ \tilde{\phi}^{-1}$ at $\phi(p)$.
- (b) Show that the Jacobian determinant of the transition function $\tilde{\psi} \circ \tilde{\phi}^{-1}$ at $\phi(p)$ is $(\det[\partial y^i / \partial x^j])^2$.

SOLUTION: Letting $q = \left(p, \sum_i a^i \frac{\partial}{\partial x^i} \Big|_p\right) \in TU$ we have by definition that

$$\begin{aligned}\tilde{\psi}^i(q) &= y^i(p) \\ \tilde{\psi}^{i+n}(q) &= \sum_j a^j \frac{\partial y^i}{\partial x^j}(p)\end{aligned}$$

for $i = 1, \dots, n$ in which case the matrix entries of the Jacobian are

$$\begin{aligned}\frac{\partial \tilde{\psi}^i}{\partial x^j}(q) &= \frac{\partial y^i}{\partial x^j}(p) \\ \frac{\partial \tilde{\psi}^i}{\partial a^j}(q) &= \frac{\partial y^i}{\partial a^j}(p) = 0\end{aligned}$$

and

$$\begin{aligned}\frac{\partial \tilde{\psi}^{i+n}}{\partial x^j}(q) &= \sum_k a^k \frac{\partial^2 y^i}{\partial x^j \partial x^k}(p) \\ \frac{\partial \tilde{\psi}^{i+n}}{\partial a^j}(q) &= \frac{\partial y^i}{\partial x^j}(p)\end{aligned}$$

So that the Jacobian matrix has the form

$$J = \begin{bmatrix} A & 0 \\ B & A \end{bmatrix}$$

where $A = \left[\frac{\partial y^i}{\partial x^j}(p)\right]$ is the Jacobian matrix of the transition function $\psi \circ \phi^{-1}$ and $B = \left[\sum_k a^k \frac{\partial^2 y^i}{\partial x^j \partial x^k}(p)\right]$. By a lemma from linear algebra, we have that $\det J = (\det A)^2$. For completeness we prove this below:

Lemma 3.2. *If a square matrix has the form*

$$M = \begin{bmatrix} A & 0 \\ B & C \end{bmatrix}$$

where A and C are square matrices, then $\det M = \det A \det C$.

Proof. Say $A = [a_j^i]$ and $M = [m_j^i]$ and that A has size $n \times n$ and C has size $m \times m$. Then

$$\det M = \sum_{i=1}^{n+m} (-1)^{1+i} m_i^1 \det M_i^1 = \sum_{i=1}^n (-1)^{1+i} a_i^1 \det M_i^1 \quad (3.11)$$

where M_i^1 is the matrix M with the first row and i -th column removed. Then M_i^1 is of the form

$$\begin{bmatrix} A_i^1 & 0 \\ B_i & C \end{bmatrix}$$

where B_i is B with the i -th column removed. By induction on the size of A , $\det M_i^1 = \det A_i^1 \det C$ —the base case is trivial. Thus we have that

$$\sum_{i=1}^n (-1)^{1+i} a_i^1 \det M_i^1 = \sum_{i=1}^n (-1)^{1+i} a_i^1 \det A_i^1 \det C = \det A \det C \quad (3.12)$$

By combining equations (3.11) and (3.12) we have the result. $\square$

PROBLEM 12.3: Smoothness of scalar multiplication

Prove that if s is a C^∞ section of a C^∞ vector bundle $\pi : E \rightarrow M$ and f is a C^∞ real-valued function on M then the product $fs : M \rightarrow E$ defined by

$$(fs)(p) = f(p)s(p) \in E_p, \quad p \in M$$

is a C^∞ section of E .

SOLUTION: Let $p \in M$ and let $U \subset M$ be a trivializing open set containing p with trivialization $\phi : \pi^{-1}(U) \rightarrow U \times \mathbb{R}^n$. Because s is smooth, $\phi \circ s : M \rightarrow U \times \mathbb{R}^n$ is smooth so that if

$$(\phi \circ s)(p) = \left(p, \sum_i c^i(p) e_i \right)$$

then the functions $c^i : M \rightarrow \mathbb{R}$ are all smooth. Then, because ϕ is linear at each p , we have that

$$\begin{aligned} (\phi \circ fs)(p) &= \phi(f(p)s(p)) \\ &= f(p)\phi(s(p)) && \text{(by linearity)} \\ &= f(p) \cdot \left(p, \sum_i c^i(p) e_i \right) \\ &= \left(p, \sum_i f(p) c^i(p) e_i \right) \end{aligned}$$

Note that $p \mapsto f(p)c^i(p)$ is smooth since both f and c^i are. Thus we have that, $\phi \circ fs$ is smooth at p and since ϕ is a diffeomorphism, fs is smooth at p . Since p was arbitrary, fs is smooth.

PROBLEM 12.4: Coefficients relative to a smooth frame

Let $\pi : E \rightarrow M$ be a C^∞ vector bundle and $s_1, \dots, s_n$ a C^∞ frame for E over an open set U in M . Then every $e \in \pi^{-1}(U)$ can be written uniquely as a linear combination

$$e = \sum_{i=1}^r c^i(e) s_i(p), \quad p = \pi(e) \in U \quad (3.13)$$

Prove that $c^i : \pi^{-1}(U) \rightarrow \mathbb{R}$ is C^∞ for $i = 1, \dots, r$.

SOLUTION: Let $\phi : E|_V \xrightarrow{\cong} V \times \mathbb{R}$ be a trivialization over $e \in V \subset U$ with $t_1, \dots, t_n$ its frame so that

$$e = \sum_{j=1}^r b^j(e) t_j(p) \quad (3.14)$$

for functions $b^j : E|_V \rightarrow \mathbb{R}$. Then we have that

$$\begin{aligned} \phi(e) &= \phi \left(\sum_{j=1}^r b^j(e) t_j(p) \right) \\ &= \sum_{j=1}^r b^j(e) \phi(t_j(p)) \quad (\text{by linearity at } p) \\ &= \sum_{j=1}^r b^j(e)(p, e_j) \quad (\text{by definition of the } t_j) \\ &= \left(p, \sum_{j=1}^r b^j(e) e_j \right) \end{aligned}$$

which merely says that the b^j are the fiber coordinates of ϕ and thus are smooth by Proposition 6.16.

Now given the frame $s_1, \dots, s_n$ restricted to $E|_V$ we have that

$$t_j = \sum_{i=1}^r a_j^i s_i$$

for some functions $a_j^i : M \rightarrow \mathbb{R}$. Because the t_j are smooth and the s_i are a smooth frame, the a_j^i are smooth by Proposition 12.12. By equations (3.13)

and (3.14), e is both $\sum b^j(e)t_j(p)$ and $\sum c^i(e)s_i(p)$ so that

$$\begin{aligned}\sum_{i=1}^r c^i(e)s_i(p) &= \sum_{j=1}^r b^j(e)t_j(p) \\ &= \sum_{i,j}^r b^j(e)a_j^i(p)s_i(p)\end{aligned}$$

so that by comparing coefficients we have

$$c^i(e) = \sum_{j=1}^r b^j(e)a_j^i(p) = \sum_{j=1}^r b^j(e) \cdot (a_j^i \circ \pi)(e)$$

Since we have already shown that the b^j and a_j^i are smooth and π is smooth by definition, we have that the c^i are smooth at e . Since $e \in U$ was arbitrary, the $c^i : \pi^{-1}(U) \rightarrow \mathbb{R}$ are smooth.

A DIGRESSION ON BUMP FUNCTION CONSTRUCTION

In Figure 3.4, the final plot shows the bump function centered at c that is identically 1 within $]c - a, c + a[$ and supported in $[c - b, c + b]$. Its formula is

$$\text{bump}(c, a, b) = x \mapsto 1 - g\left(\frac{(x - c)^2 - a^2}{b^2 - a^2}\right)$$

and in the plot we have $c = 0.5, a = 0.3, b = 0.75$. Figure 3.5 shows the bump function centered at 0 in $\mathbb{R}^2$ that is 1 within the disc $\|x\| < 2$ and has support $\|x\| \leq 4$.

PROBLEM 13.1: Support of a finite sum

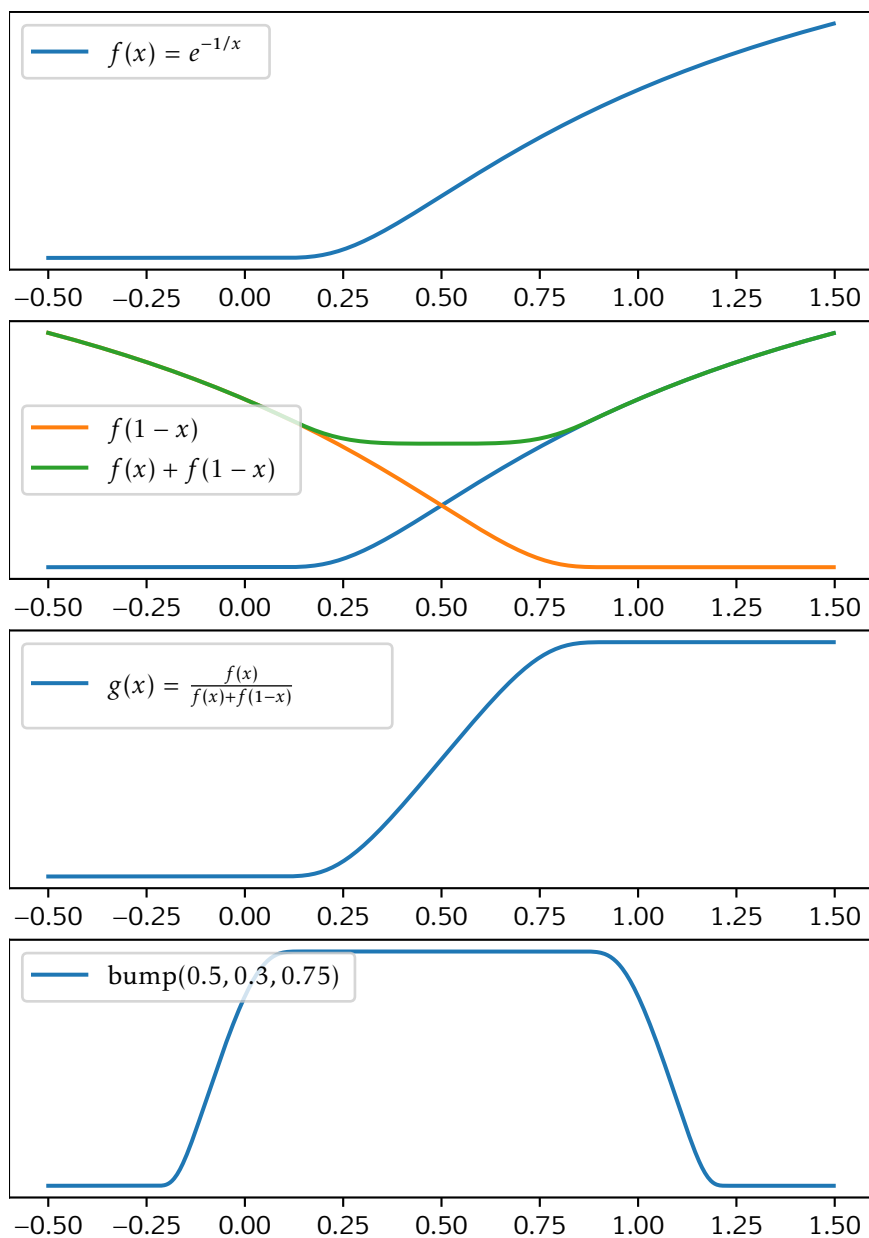
Prove that if $\rho_1, \dots, \rho_m$ are real-valued functions on a manifold M , then

$$\text{supp}\left(\sum \rho_i\right) \subset \bigcup \text{supp } \rho_i$$

SOLUTION: Say $x \in \text{supp}(\sum \rho_i)$ and let U be a neighborhood of x . Thus U intersects a non-zero value of $\sum \rho_i$ so that at least one must be non-zero. Thus U intersects $\bigcup \rho_i^{-1}(\mathbb{R}^\times)$. Since U was arbitrary, $x \in \text{cl}\left(\bigcup \rho_i^{-1}(\mathbb{R}^\times)\right)$. By Exercise A.53,

$$\text{cl}\left(\bigcup \rho_i^{-1}(\mathbb{R}^\times)\right) = \bigcup \text{cl}(\rho_i^{-1}(\mathbb{R}^\times)) = \bigcup \text{supp } \rho_i$$

which proves the claim.

Figure 3.4: Construction of the C^∞ bump function.

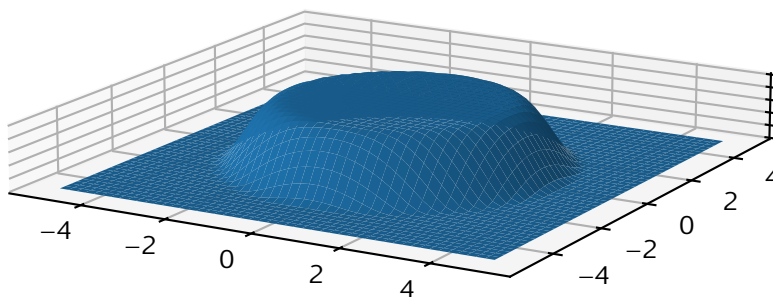


Figure 3.5: Bump function on $\mathbb{R}^2$ centered at 0.

EXERCISE A.53(A): Closure of a finite union.

Let A and B be subsets of a topological space S . Show that $\text{cl}(A \cup B) = \text{cl}(A) \cup \text{cl}(B)$.

SOLUTION: Since $A \subset \text{cl}(A)$ and $B \subset \text{cl}(B)$ we have that $A \cup B \subset \text{cl}(A) \cup \text{cl}(B)$. Since $\text{cl}(A) \cup \text{cl}(B)$ is closed, $\text{cl}(A \cup B) \subset \text{cl}(A) \cup \text{cl}(B)$ since the closure is the smallest closed set that contains $A \cup B$.

Conversely let $x \in \text{cl}(A) \cup \text{cl}(B)$ and U a neighborhood of x . If $x \in \text{cl}(A)$ then U intersects A so that it intersects $A \cup B$ in which case $x \in \text{cl}(A \cup B)$. Similarly if $x \in \text{cl}(B)$ then also $x \in \text{cl}(A \cup B)$. Thus, $\text{cl}(A) \cup \text{cl}(B) \subset \text{cl}(A \cup B)$.

PROBLEM 13.2: Locally finite family and compact set

Let $\{A_\alpha\}$ be a locally finite family of subsets of a topological space S . Show that every compact set K in S has a neighborhood W that intersects only finitely many of the A_α .

SOLUTION: Because the A_α are locally finite, each point $p \in K$ has a neighborhood U_p that intersects the A_α finitely. Since K is compact, the U_p have a finite subcover U_{p_i} for some points $p_1, \dots, p_n$ each of which intersects finitely with some sets $A_{i,j}$, for $j = 1, \dots, n_i$. Let $W = \bigcup_i U_{p_i}$ in which case

we have

$$\begin{aligned}
 W \cap \bigcup_{\alpha} A_{\alpha} &= \bigcup_i U_{p_i} \cap \bigcup_{\alpha} A_{\alpha} \\
 &= \bigcup_i \left(U_{p_i} \cap \bigcup_{\alpha} A_{\alpha} \right) \\
 &= \bigcup_i \left(U_{p_i} \cap \bigcup_j A_{i,j} \right) \\
 &\subset \bigcup_i U_{p_i} \cap \bigcup_{i,j} A_{i,j} \\
 &= W \cap \bigcup_{i,j} A_{i,j}
 \end{aligned}$$

so that W intersects only with the finitely many $A_{i,j}$.

PROBLEM 13.3: Smooth Urysohn Lemma

- Let A and B be two disjoint closed sets in a manifold M . Find a C^{∞} function f on M such that f is identically 1 on A and identically 0 on B .
- Let A be a closed subset and U an open subset of a manifold M . Show that there is a C^{∞} function f on M such that f is identically 1 on A and $\text{supp } f \subset U$.

SOLUTION:

- Since A and B are disjoint, their complements $M \setminus A$ and $M \setminus B$ form an open cover of M because for $x \in M$ if $x \in A$ then $x \notin B$ so that $x \in M \setminus B$. Otherwise $x \notin A$ so that $x \in M \setminus A$. Thus $M \subset (M \setminus A) \cup (M \setminus B)$ because x was arbitrary.

Thus by Theorem 13.7, there is a C^{∞} partition of unity $\rho_{M \setminus A}, \rho_{M \setminus B}$ subordinate to $\{M \setminus A, M \setminus B\}$. Let $f = \rho_{M \setminus B}$. If $x \in A$ then $x \notin M \setminus A$ so that $\rho_{M \setminus A}(x) = 0$ since $\rho_{M \setminus A}(x)$ is supported in $M \setminus A$. Then

$$\begin{aligned}
 1 &= \rho_{M \setminus A}(x) + \rho_{M \setminus B}(x) \\
 &= \rho_{M \setminus B}(x) \\
 &= f(x)
 \end{aligned}$$

so that f is 1 on A . If $x \in B$ then $x \notin M \setminus B$ which contains the support of $f = \rho_{M \setminus B}$ so that $f(x) = 0$. In fact we have proved the slightly stronger claim that not only is $f = 0$ on B but that the support of f is contained in the complement $M \setminus B$.

- $M \setminus U$ is closed and disjoint from A so that by the previous problem there is a function f that is identically 1 on A and whose support is in $M \setminus (M \setminus U) = U$.

PROBLEM 13.4: Support of the pullback of a function

Let $F : N \rightarrow M$ be a C^∞ map of manifolds and $h : M \rightarrow \mathbb{R}$ a C^∞ real-valued function. Prove that $\text{supp } F^*h \subset F^{-1}(\text{supp } h)$.

SOLUTION: Suppose $x \in (F^*h)^{-1}(\mathbb{R}^\times)$ so that $h(F(x)) \neq 0$. Then $F(x) \in h^{-1}(\mathbb{R}^\times) \subset \text{supp } h$ so that $x \in F^{-1}(\text{supp } h)$. Thus we have shown that $(F^*h)^{-1}(\mathbb{R}^\times) \subset F^{-1}(\text{supp } h)$. Note that $F^{-1}(\text{supp } h)$ is closed since F is continuous. Since $\text{supp } F^*h$ is the smallest closed subset that contains $(F^*h)^{-1}(\mathbb{R}^\times)$, we have that $\text{supp } F^*h \subset F^{-1}(\text{supp } h)$, proving the claim.

PROBLEM 13.5: Support of the pullback by a projection

Let $f : M \rightarrow \mathbb{R}$ be a C^∞ function on a manifold M . If N is another manifold and $\pi : M \times N \rightarrow M$ is the projection onto the first factor, prove that

$$\text{supp}(\pi^*f) = (\text{supp } f) \times N$$

SOLUTION: First assume that $(\pi^*f)(a, b) \neq 0$ which is to say that $f(a) \neq 0$. Thus $(a, b) \in (\text{supp } f) \times N$. Since $(\text{supp } f) \times N$ is closed in the product topology, and (a, b) were arbitrary non-zero values of π^*f , we have that $\text{supp}(\pi^*f) \subset (\text{supp } f) \times N$ since $\text{supp}(\pi^*f)$ is the smallest closed set that contains the non-zero values of π^*f .

Now suppose that $(a, b) \in (\text{supp } f) \times N$ and let U be an arbitrary neighborhood of (a, b) . Then there exist neighborhoods U_M and U_N of a and b respectively such that $U_M \times U_N \subset U$. Since $a \in \text{supp } f$, U_M must intersect with the non-zero values of f . Thus we can find an $x \in U_M$ such that $f(x) \neq 0$ in which case $(\pi^*f)(x, b) = f(x) \neq 0$. Thus $U_M \times U_N$ intersects with the non-zero values of π^*f and so U does so as well. Since U was an arbitrary neighborhood, $(a, b) \in \text{supp}(\pi^*f)$ and since (a, b) were arbitrary, $(\text{supp } f) \times N \subset \text{supp}(\pi^*f)$.

PROBLEM 13.6: Pullback of a partition of unity

Suppose $\{\rho_\alpha\}$ is a partition of unity on a manifold M subordinate to an open cover $\{U_\alpha\}$ of M and $F : N \rightarrow M$ is a C^∞ map. Prove that

- the collection of supports $\{\text{supp } F^*\rho_\alpha\}$ is locally finite;
- the collection of functions $\{F^*\rho_\alpha\}$ is a partition of unity on N subordinate to the open cover $\{F^{-1}(U_\alpha)\}$.

SOLUTION:

- Let $q \in N$. Because the $\{\text{supp } \rho_\alpha\}$ are locally finite, $F(q) \in M$ has a neighborhood U with finite intersection $\{\text{supp } \rho_i\} \subset \{\text{supp } \rho_\alpha\}$. Thus

if $x \in F^{-1}(U)$ we have that $F(x) \in U$ and thus

$$\begin{aligned} 1 &= \sum_{\alpha} \rho_{\alpha}(F(x)) \\ &= \sum_i \rho_i(F(x)) \\ &= \sum_i F^* \rho_i(x) \end{aligned}$$

so that $x \in \text{supp}(\sum F^* \rho_i)$. By Lemma 13.5, $x \in \bigcup \text{supp } F^* \rho_i$. Since $x \in F^{-1}(U)$ was arbitrary, $F^{-1}(U)$ intersects finitely with the $\{\text{supp } F^* \rho_i\}$ of the $\{\text{supp } F^* \rho_{\alpha}\}$. Because $q \in N$ was arbitrary, $\{\text{supp } F^* \rho_{\alpha}\}$ is locally finite.

- For any $q \in N$ we have that $1 = \sum \rho_{\alpha}(F(q)) = \sum F^* \rho_{\alpha}(q)$ so that $\sum F^* \rho_{\alpha} = 1$. Thus $\{F^* \rho_{\alpha}\}$ is a partition of unity. By problem 13.4, we have that $\text{supp } F^* \rho_{\alpha} \subset F^{-1}(\text{supp } \rho_{\alpha}) \subset F^{-1}(U_{\alpha})$ since $\text{supp } \rho_{\alpha}$ is subordinate to U_{α} . Thus $\text{supp } F^* \rho_{\alpha}$ is subordinate to $F^{-1}(U_{\alpha})$.

PROBLEM 13.7: Closure of a locally finite union

If $\{A_{\alpha}\}$ is a locally finite collection of subsets in a topological space, then

$$\overline{\bigcup A_{\alpha}} = \bigcup \overline{A_{\alpha}}$$

SOLUTION: The inclusion $\bigcup \overline{A_{\alpha}} \subset \overline{\bigcup A_{\alpha}}$ is always true since if $x \in \bigcup \overline{A_{\alpha}}$ and U is a neighborhood of x , then since $x \in \overline{A_{\alpha}}$ for some α , U intersects that A_{α} . But then U intersects $\bigcup A_{\alpha}$ so that $x \in \overline{\bigcup A_{\alpha}}$ since U was arbitrary.

For the other direction, let $x \in \overline{\bigcup A_{\alpha}}$. Since $\{A_{\alpha}\}$ is locally finite, x has a neighborhood V that intersects finitely many $\{A_i\} \subset \{A_{\alpha}\}$. Now given an arbitrary neighborhood U of x , $U \cap V$ is also a neighborhood of x so that it intersects the $\bigcup A_{\alpha}$ since x is in the closure. Thus we have that

$$\begin{aligned} \emptyset &\neq U \cap V \cap \bigcup A_{\alpha} \\ &= U \cap V \cap \bigcup A_i \\ &\subset U \cap \bigcup A_i \end{aligned}$$

Thus, since $U \cap \bigcup A_i$ is non-empty, it intersects finitely with the $\{A_i\}$. Since U was arbitrary, we have that $x \in \overline{\bigcup A_i} = \bigcup \overline{A_i}$ by problem A.53 since the $\{A_i\}$ are finite. Since each A_i is an A_{α} , we have that $\bigcup \overline{A_i} \subset \bigcup \overline{A_{\alpha}}$ so that $x \in \bigcup \overline{A_{\alpha}}$.

PROBLEM 14.1: Equality of vector fields

Show that two C^{∞} vector fields X and Y on a manifold M are equal if and only if for every C^{∞} function on M , we have $Xf = Yf$.

SOLUTION: That the condition is necessary is trivial so assume that $Xf = Yf$ for any $f : M \rightarrow \mathbb{R}$. Let $p \in M$ and $(U, \phi) = (U, x^1, \dots, x^n)$ a chart around p so that $X_p = \sum a^i \frac{\partial}{\partial x^i} |_p$ and $Y_p = \sum b^i \frac{\partial}{\partial x^i} |_p$. By Proposition 13.2 we can extend each x^i to a function $\tilde{x}^i$ on all of M that agrees with x^i in a neighborhood of p . Thus we have that

$$a^i = X_p \tilde{x}^i = Y_p \tilde{x}^i = b^i$$

so that $X_p = Y_p$. Since p was arbitrary, $X = Y$.

PROBLEM 14.2: Vector field on an odd sphere

Let $x^1, y^1, \dots, x^n, y^n$ be the standard coordinates on $\mathbb{R}^{2n}$. The unit sphere S^{2n-1} in $\mathbb{R}^{2n}$ is defined by the equation $\sum_{i=1}^n (x^i)^2 + (y^i)^2 = 1$. Show that

$$X = \sum_{i=1}^n -y^i \frac{\partial}{\partial x^i} + x^i \frac{\partial}{\partial y^i}$$

is a nowhere-vanishing smooth vector field on S^{2n-1} .

SOLUTION: We have that $\sum -y^i x^i + x^i y^i = 0$ so that this is a tangent vector at the point (x^i, y^i) by problem 11.1. To find the integral curves of this flow, consider a point $(a^i, b^i) \in S^{2n-1}$ and let

$$c^i = \sqrt{(a^i)^2 + (b^i)^2}$$

Because $(a^i)^2 \leq (a^i)^2 + (b^i)^2 = (c^i)^2$, we have that $-c^i \leq a^i \leq c^i$. Because $c^i \cos t$ ranges from $-c^i$ to c^i there must exist a t_0^i such that $c^i \cos t_0^i = a^i$ by the intermediate value theorem. In that case we have that

$$\begin{aligned} (b^i)^2 &= (a^i)^2 + (b^i)^2 - (a^i)^2 \\ &= (c^i)^2 - (c^i \cos t_0^i)^2 \\ &= (c^i)^2 (1 - \cos^2 t_0^i) \\ &= (c^i)^2 \sin^2 t_0^i \end{aligned}$$

so that $b^i = c^i \sin t_0^i$. In that case the curve

$$c(t) = (c^i \cos(t - t_0^i), c^i \sin(t - t_0^i))$$

satisfies

$$c(0) = (c^i \cos t_0^i, c^i \sin t_0^i) = (a^i, b^i)$$

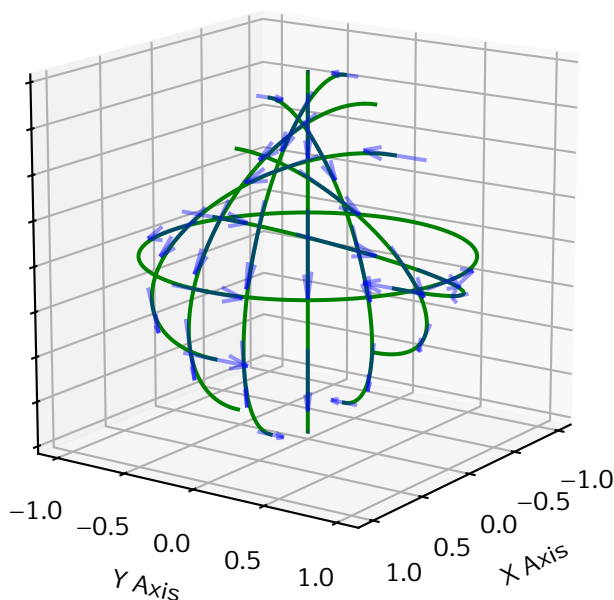


Figure 3.6: Problem 14.2

and

$$\begin{aligned} c'(t) &= \sum_{i=1}^n -c^i \sin(t - t_0^i) \frac{\partial}{\partial x^i} + c^i \cos(t - t_0^i) \frac{\partial}{\partial y^i} \\ &= \sum_{i=1}^n -y^i(c(t)) \frac{\partial}{\partial x^i} + x^i(c(t)) \frac{\partial}{\partial y^i} \end{aligned}$$

and so is the unique flow at (a^i, b^i) . Each integral curve thus corresponds uniquely to choosing the set of c^i which satisfy

$$\sum_{i=1}^n (c^i)^2 = \sum_{i=1}^n (a^i)^2 + (b^i)^2 = 1$$

so that the space of curves is itself a sphere with dimension $n - 1$.

Figure 3.6 shows this vector field and its integral curves for S^3 in the chart $\{(x, y, z, w) \mid w > 0\}$. The circle going around the “equator” of this chart is the curve $(\cos t, \sin t, 0, 0)$. The line going straight down the z -axis is the curve $(0, 0, \cos t, \sin t)$.

PROBLEM 14.3: Maximal integral curve on a punctured line

Let M be $\mathbb{R} - \{0\}$ and let X be the vector field d/dx on M . Find the maximal integral curve of X starting at $x = 1$.

SOLUTION: The curve is $c(x) = x + 1$ defined on $] -1, \infty[$ in which case $c'(x) = d/dx$.

PROBLEM 14.4: Integral curves in the plane

Find the integral curves of the vector field

$$X_{(x,y)} = x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y} \quad \text{on } \mathbb{R}^2$$

SOLUTION: The integral curve at the point $(a, b) \in \mathbb{R}^2$ is $c(t) = (ae^t, be^{-t})$ because then we have

$$\begin{aligned} c(0) &= (a, b) \\ c'(t) &= ae^t \frac{\partial}{\partial x} - be^{-t} \frac{\partial}{\partial y} \end{aligned}$$

The graph of various integral curves of this vector field is given in Figure 3.7.

PROBLEM 14.5: Maximal integral curve in the plane

Find the maximal integral curve $c(t)$ starting at the point $(a, b) \in \mathbb{R}^2$ of the vector field $X_{(x,y)} = \partial/\partial x + x\partial/\partial y$ on $\mathbb{R}^2$.

SOLUTION: The curve is $c(t) = (t + a, \frac{1}{2}t^2 + ta + b)$ because then

$$c'(t) = \frac{\partial}{\partial x} + (t + a) \frac{\partial}{\partial y}$$

This is the hamiltonian phase space for a constant acceleration of 1. The x coordinate is velocity and y is position. The initial point a specifies the starting velocity and b specifies the starting position. Various integral curves and the vector field are graphed in Figure 3.8.

PROBLEM 14.7: Maximal integral curve

Let X be the vector field xd/dx on R . For each p in $\mathbb{R}$, find the maximal integral curve $c(t)$ of X starting at p .

SOLUTION: This is $c(t) = pe^t$ because then $c(0) = pe^0 = p$ and $c'(t) = pe^t d/dx$.

PROBLEM 14.8: Maximal integral curve

Let X be the vector field $x^2 d/dx$ on the real line R . For each $p > 0$ in R , find the maximal integral curve of X with initial point p .

SOLUTION: Following the example in the text, we have that

$$x(t) = -\frac{1}{t + C}$$

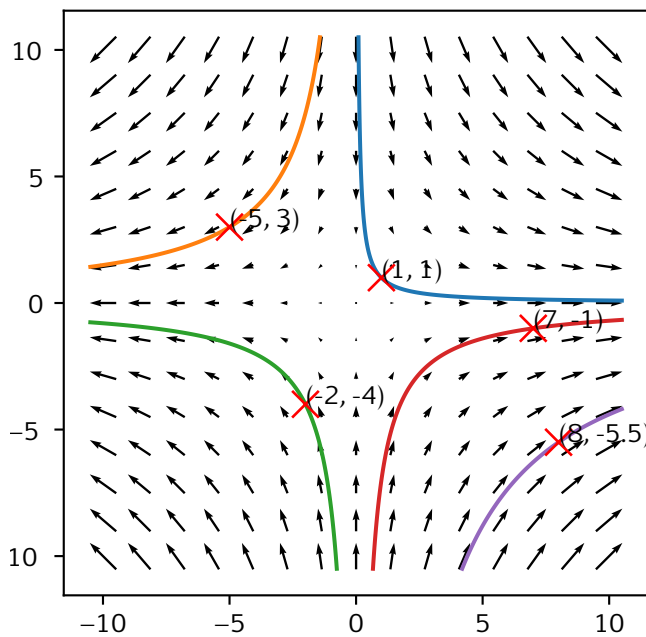


Figure 3.7: Problem 14.4

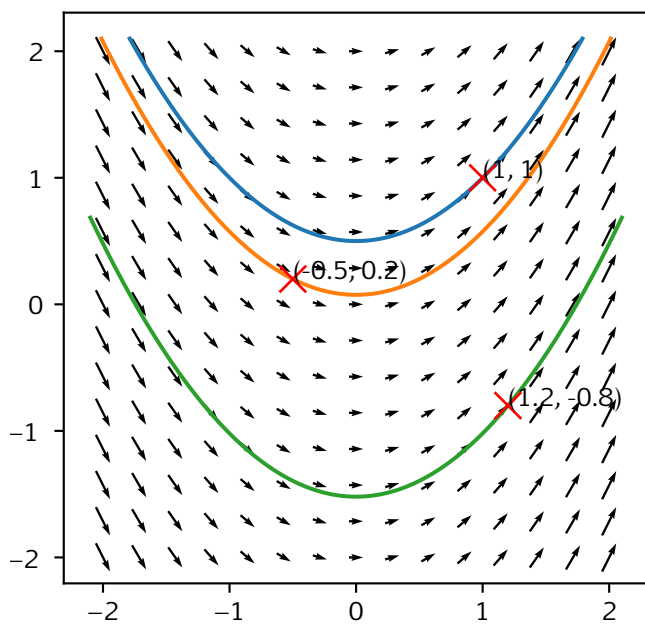


Figure 3.8: Problem 14.5

The condition $x(0) = p$ means that $p = -1/C$ so that $C = -1/p$. Thus we have the equation

$$x(t) = -\frac{1}{t - \frac{1}{p}}$$

which is defined as long as $t \neq \frac{1}{p}$. Thus the maximal interval containing 0 is $\left] \infty, \frac{1}{p} \right[$.

N.B. in the problems that follow, we identify a vector field X with a function $X : C^\infty(M) \rightarrow C^\infty(M)$ that is a derivation on products even though this correspondence will not be proved until section 19. This allows to avoid awkward calculations with X_p and calculate globally using the vector fields operation on functions.

PROBLEM 14.10: Lie bracket of vector fields

If f and g are C^∞ functions and X and Y are C^∞ vector fields on a manifold M , show that

$$[fX, gY] = fg[X, Y] + f(Xg)Y - g(Yf)X$$

SOLUTION: Apply $[fX, gY]$ to a smooth function h and use the fact that X and Y are derivations on products of functions:

$$\begin{aligned} [f \cdot X, g \cdot Y]h &= f \cdot X(g \cdot Yh) - g \cdot Y(f \cdot Xh) \\ &= f \cdot Xg \cdot Yh + f \cdot g \cdot X(Yh) - g \cdot Yf \cdot Xh - g \cdot f \cdot Y(Xh) \\ &= f \cdot g \cdot (X(Yh) - Y(Xh)) + f \cdot Xg \cdot Yh - g \cdot Yf \cdot Xh \\ &= f \cdot g \cdot [X, Y]h + f \cdot Xg \cdot Yh - g \cdot Yf \cdot Xh \\ &= (f \cdot g \cdot [X, Y] + f \cdot Xg \cdot Y - g \cdot Yf \cdot X)h \end{aligned}$$

so that $[fX, gY] = fg[X, Y] + f(Xg)Y - g(Yf)X$

PROBLEM 14.11: Lie bracket of vector fields on $\mathbb{R}^2$

Compute the Lie bracket

$$\left[-y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y}, \frac{\partial}{\partial x} \right]$$

on $\mathbb{R}^2$.

SOLUTION: If

$$\left[-y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y}, \frac{\partial}{\partial x} \right] = a \frac{\partial}{\partial x} + b \frac{\partial}{\partial y}$$

then

$$\begin{aligned}
 a &= \left[-y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y}, \frac{\partial}{\partial x} \right] x \\
 &= \left(-y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} \right) \frac{\partial x}{\partial x} - \frac{\partial}{\partial x} \left(-y \frac{\partial x}{\partial x} + x \frac{\partial x}{\partial y} \right) \\
 &= -y \frac{\partial 1}{\partial x} + x \frac{\partial 1}{\partial y} + \frac{\partial y}{\partial x} \\
 &= 0 + 0 + 0 = 0
 \end{aligned}$$

and

$$\begin{aligned}
 b &= \left[-y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y}, \frac{\partial}{\partial x} \right] y \\
 &= \left(-y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} \right) \frac{\partial y}{\partial x} - \frac{\partial}{\partial x} \left(-y \frac{\partial y}{\partial x} + x \frac{\partial y}{\partial y} \right) \\
 &= -\frac{\partial}{\partial x} x \\
 &= -1
 \end{aligned}$$

so that

$$\left[-y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y}, \frac{\partial}{\partial x} \right] = -\frac{\partial}{\partial y}$$

PROBLEM 14.12: Lie bracket in local coordinates

Consider two C^∞ vector fields X, Y on $\mathbb{R}^n$:

$$X = \sum a^i \frac{\partial}{\partial x^i} \quad Y = \sum b^i \frac{\partial}{\partial x^i}$$

where a^i, b^j are C^∞ functions on $\mathbb{R}^n$. Since $[X, Y]$ is also a C^∞ vector field on $\mathbb{R}^n$,

$$[X, Y] = \sum c^i \frac{\partial}{\partial x^i}$$

for some C^∞ functions c^i . Find the formula for c^i in terms of a^i and b^i .

SOLUTION:

$$\begin{aligned}
 c^i &= [X, Y]x^i \\
 &= \left(\sum_j a^j \frac{\partial}{\partial x^j} \right) \left(\sum_j b^j \frac{\partial x^i}{\partial x^j} \right) - \left(\sum_j b^j \frac{\partial}{\partial x^j} \right) \left(\sum_j a^j \frac{\partial x^i}{\partial x^j} \right) \\
 &= \left(\sum_j a^j \frac{\partial}{\partial x^j} \right) b^i - \left(\sum_j b^j \frac{\partial}{\partial x^j} \right) a^i \\
 &= \sum_j a^j \frac{\partial b^i}{\partial x^j} - b^j \frac{\partial a^i}{\partial x^j}
 \end{aligned}$$

PROBLEM 14.13: Vector field under a diffeomorphism

Let $F : N \rightarrow M$ be a C^∞ diffeomorphism of manifolds. Prove that if g is a C^∞ function and X a C^∞ vector field on N , then

$$F_*(gX) = (g \circ F^{-1})F_*X$$

SOLUTION: Given an arbitrary C^∞ function f on M , we have by definition that $F_*(gX)f = gX(f \circ F)$. Applying this function to an arbitrary point $p \in N$, let $q = F(p)$ in which case we have that

$$\begin{aligned}
 (F_*(gX)f)(p) &= g(p)X_p(f \circ F) \\
 &= g(F^{-1}(q))F_{*,p}(X_p)(f) \\
 &= g(F^{-1}(q))F_*(X)_q(f) \\
 &= ((g \circ F^{-1})F_*(X)f)(q)
 \end{aligned}$$

so that $F_*(gX)f = (g \circ F^{-1})F_*(X)f$. Since f was arbitrary, we have that $F_*(gX) = (g \circ F^{-1})F_*(X)$ by Exercise 14.1.

PROBLEM 14.14: Lie bracket under a diffeomorphism

Let $F : N \rightarrow M$ be a C^∞ diffeomorphism of manifolds. Prove that if X, Y are C^∞ vector fields on N , then

$$F_*[X, Y] = [F_*X, F_*Y]$$

SOLUTION: The vector fields X and F_*X are F -related and similarly for Y and F_*Y , in which case $[X, Y]$ and $[F_*X, F_*Y]$ are F -related by Proposition 14.17. Thus $F_*[X, Y] = [F_*X, F_*Y]$ by the definition of F -relatedness.

Chapter 4

Lie Groups and Lie Algebras

PROBLEM 15.1: Matrix exponential

For $X \in \mathbb{R}^{n \times n}$, define the partial sum $s_m = \sum_{k=0}^m X^k / k!$.

(a) Show that for $l \geq m$,

$$\|s_l - s_m\| \leq \sum_{k=m+1}^l \frac{\|X\|^k}{k!}$$

(b) Conclude that s_m is a Cauchy sequence in $\mathbb{R}^{n \times n}$ and therefore converges to a matrix which we denote by e^X .

SOLUTION:

•

$$\begin{aligned} \|s_l - s_m\| &= \left\| \sum_{k=0}^l \frac{X^k}{k!} - \sum_{k=0}^m \frac{X^k}{k!} \right\| \\ &= \left\| \sum_{k=m+1}^l \frac{X^k}{k!} + \sum_{k=0}^m \frac{X^k}{k!} - \sum_{k=0}^m \frac{X^k}{k!} \right\| \\ &= \left\| \sum_{k=m+1}^l \frac{X^k}{k!} \right\| \\ &\leq \sum_{k=m+1}^l \left\| \frac{X^k}{k!} \right\| \end{aligned}$$

• Since

$$\sum_{k=0}^{\infty} \frac{\|X\|^k}{k!} = e^{\|X\|}$$

we have that for any ϵ , there is an N such that for any $m \geq n \geq N$

$$\sum_{k=m+1}^l \frac{\|X\|^k}{k!} < \epsilon$$

But this is simply $\|s_l - s_m\|$ so that s_m is a Cauchy sequence.

PROBLEM 15.2: Product rule for matrix-valued functions

Let $]a, b[$ be an open interval in $\mathbb{R}$. Suppose $A :]a, b[\rightarrow \mathbb{R}^{m \times n}$ and $B :]a, b[\rightarrow \mathbb{R}^{n \times p}$ are $m \times n$ and $n \times p$ matrices respectively whose entries are differentiable functions of $t \in]a, b[$. Prove that for $t \in]a, b[$,

$$\frac{d}{dt} A(t)B(t) = A'(t)B(t) + A(t)B'(t)$$

where $A'(t) = (dA/dt)(t)$ and $B'(t) = (dB/dt)(t)$.

SOLUTION: Say $A = [a_j^i]$ and $B = [b_j^i]$ where a_j^i and b_j^i are functions of $t \in]a, b[$. Then

$$\begin{aligned} \frac{d}{dt} A \cdot B &= \left[\frac{d}{dt} \sum_k a_k^i b_j^k \right] && \text{(by matrix multiplication)} \\ &= \left[\sum_k \frac{da_k^i}{dt} b_j^k + a_k^i \frac{db_j^k}{dt} \right] && \text{(by the product rule)} \\ &= \left[\sum_k \frac{da_k^i}{dt} b_j^k \right] + \left[\sum_k a_k^i \frac{db_j^k}{dt} \right] \\ &= A'(t)B(t) + A(t)B'(t) && \text{(by matrix multiplication)} \end{aligned}$$

PROBLEM 15.5: Differential of the multiplication map

Let G be a Lie group with multiplication map $\mu : G \times G \rightarrow G$, and let $l_a : G \rightarrow G$ and $r_b : G \rightarrow G$ be left and right multiplication by a and $b \in G$, respectively. Show that the differential of μ at $(a, b) \in G \times G$ is

$$\mu_{*,(a,b)}(X_a, Y_b) = (r_b)_* X_a + (l_a)_* Y_b$$

for $X_a \in T_a(G)$ and $Y_b \in T_b(G)$.

SOLUTION: Let $c_a(t) : \mathbb{R} \rightarrow G$ and $c_b(t) : \mathbb{R} \rightarrow G$ be curves at a and b respectively such that $c'_a(0) = X_a$ and $c'_b(0) = Y_b$. Then $\bar{c}_a(t) = (c_a(t), b)$ and $\bar{c}_b(t) = (a, c_b(t))$ are both curves at (a, b) such that, by Exercise 8.7,

$$\bar{c}_a'(0) = (c'_a(0), 0) = (X_a, 0) \quad (4.1)$$

since b is constant and similarly $\bar{c}_b'(t) = (0, c'_b(t)) = (0, Y_b)$. We also have that

$$(\mu \circ \bar{c}_a)(t) = \mu(c_a(t), b) = r_b(c_a(t)) = (r_b \circ c_a)(t) \quad (4.2)$$

and similarly $\mu \circ \overline{c}_b = l_a \circ c_b$. Thus we can calculate

$$\begin{aligned}
 \mu_{*,(a,b)}(X_a, 0) &= \mu_{*,(a,b)}(\overline{c}_a'(0)) && \text{(by (4.1))} \\
 &= (\mu_{*,(a,b)} \circ (\overline{c}_a)_*) \left(\frac{d}{dt} \Big|_0 \right) \\
 &= (r_b)_* \circ (c_a)_* \left(\frac{d}{dt} \Big|_0 \right) && \text{(by (4.2) and functoriality)} \\
 &= (r_b)_*(c'_a(0)) \\
 &= (r_b)_*(X_a)
 \end{aligned}$$

and similarly $\mu_{*,(a,b)}(0, Y_b) = (l_a)_*(Y_b)$. Thus finally we have

$$\mu_{*,(a,b)}(X_a, Y_b) = \mu_{*,(a,b)}(X_a, 0) + \mu_{*,(a,b)}(0, Y_b) = (r_b)_*X_a + (l_a)_*Y_b$$

PROBLEM 15.6: Differential of the inverse map

Let G be a Lie group with multiplication map $\mu : G \times G \rightarrow G$, inverse map $\iota : G \rightarrow G$ and identity element e . Show that the differential of the inverse map at $a \in G$,

$$i_{*,a} : T_a G \rightarrow T_{a^{-1}} G$$

is given by

$$i_{*,a}(Y_a) = -(r_{a^{-1}})_{*,e}(l_{a^{-1}})_{*,a}Y_a$$

SOLUTION: Let $c(t) : \mathbb{R} \rightarrow G$ be a curve at $a \in G$ such that $c'(0) = Y_a$ and define the curve $\bar{c} : \mathbb{R} \rightarrow G \times G$ as $\bar{c}(t) = (c(t), (\iota \circ c)(t))$. Then

$$\bar{c}'(0) = (c'(0), \iota_{*,a}(c'(0))) = (Y_a, i_{*,a}Y_a)$$

and since $\mu \circ \bar{c}(t) = \mu(c(t), \iota(c(t))) = e$ is constant, we have

$$\begin{aligned}
 0 &= (\mu \circ \bar{c})'(0) \\
 &= \mu_{*,(a,a^{-1})} \circ \bar{c}_* \left(\frac{d}{dt} \Big|_0 \right) \\
 &= \mu_{*,(a,a^{-1})}(\bar{c}'(0)) \\
 &= \mu_{*,(a,a^{-1})}(Y_a, i_{*,a}Y_a) \\
 &= (r_{a^{-1}})_*Y_a + (l_a)_*i_{*,a}Y_a \quad \text{(by Exercise 15.5)}
 \end{aligned}$$

so that

$$(l_a)_*i_{*,a}Y_a = -(r_{a^{-1}})_*Y_a$$

and therefore

$$\begin{aligned}
 i_{*,a}Y_a &= -(l_a)_*^{-1}(r_{a^{-1}})_*Y_a \\
 &= -(l_a^{-1})_*(r_{a^{-1}})_*Y_a \quad \text{(by functoriality)} \\
 &= -(l_{a^{-1}})_*(r_{a^{-1}})_*Y_a \quad \text{(definition of left multiplication)}
 \end{aligned}$$

On the other hand, by using the curve $\bar{c}(t) = ((\iota \circ c)(t), c(t))$ we get the equation

$$0 = \mu_{*,(a^{-1},a)}(\iota_{*,a}Y_a, Y_a) = (r_a)_* \iota_{*,a}Y_a + (l_{a^{-1}})_*Y_a$$

which yields

$$i_{*,a}Y_a = -(r_{a^{-1}})_*(l_{a^{-1}})_*Y_a$$

PROBLEM 15.7: Differential of the determinant map at A

Show that the differential of the determinant map $\det : \mathrm{GL}(n, \mathbb{R}) \rightarrow \mathbb{R}$ at $A \in \mathrm{GL}(n, \mathbb{R})$ is given by

$$(\det)_{*,A}(AX) = \det A \cdot \mathrm{tr} X \quad \text{for } X \in \mathbb{R}^{n \times n}$$

SOLUTION: First just a note that the general formula is

$$(\det)_{*,A}(X) = \det A \mathrm{tr}(A^{-1}X)$$

Consider the curve $c(t) = Ae^{tX}$ which satisfies $c(0) = A$ and $c'(0) = AX$ since $c'(t) = AXe^{tX}$. Then

$$\begin{aligned} (\det)_{*,A}(AX) &= (\det)_{*,A}(c'(0)) \\ &= \left. \frac{d}{dt} \right|_0 \det(c(t)) && \text{(Proposition 8.18)} \\ &= \left. \frac{d}{dt} \right|_0 \det(Ae^{tX}) \\ &= \left. \frac{d}{dt} \right|_0 \det(A) \det(e^{tX}) \\ &= \det(A) \left. \frac{d}{dt} \right|_0 \det(e^{tX}) \\ &= \det(A) \left. \frac{d}{dt} \right|_0 e^{t \mathrm{tr} X} && \text{(Proposition 15.20)} \\ &= \det(A) \mathrm{tr} X \left. e^{t \mathrm{tr} X} \right|_0 \\ &= \det(A) \mathrm{tr} X \end{aligned}$$

PROBLEM 15.8: Special linear group

Use Exercise 15.7 to show that 1 is a regular value of the determinant map. This gives a quick proof that the special linear group $\mathrm{SL}(n, \mathbb{R})$ is a regular submanifold of $\mathrm{GL}(n, \mathbb{R})$.

SOLUTION: We must show that if $\det A = 1$ then $(\det)_{*,A} : T_A \mathrm{GL}(n, \mathbb{R}) \rightarrow T_1 \mathbb{R}$ is surjective on $T_1 \mathbb{R} = \mathbb{R}$. From the previous problem, let

$$X = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

which is to say, 1 in the first row and column and 0 otherwise. Then

$$(\det)_{*,A}(AX) = \det A \cdot \operatorname{tr} X = 1 \cdot 1 = 1$$

so that $(\det)_{*,A}$ is surjective.

Chapter 5

Differential Forms

PROBLEM 17.1: A 1-form on $\mathbb{R}^2 - \{(0, 0)\}$

Denote the standard coordinates on $\mathbb{R}^2$ by x, y and let

$$X = -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} \quad \text{and} \quad Y = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}$$

be vector fields on $\mathbb{R}^2$. Find a 1-form ω on $\mathbb{R}^2 - \{(0, 0)\}$ such that $\omega(X) = 1$ and $\omega(Y) = 0$.

SOLUTION: Let

$$\omega = \frac{-ydx + xdy}{x^2 + y^2}$$

Then

$$\begin{aligned} \omega(X) &= \frac{-ydx + xdy}{x^2 + y^2} \left(-y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} \right) \\ &= \frac{-y \cdot -y + x \cdot x}{x^2 + y^2} \\ &= \frac{y^2 + x^2}{x^2 + y^2} \\ &= 1 \end{aligned}$$

and

$$\begin{aligned} \omega(Y) &= \frac{-ydx + xdy}{x^2 + y^2} \left(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right) \\ &= \frac{-y \cdot x + x \cdot y}{x^2 + y^2} \\ &= 0 \end{aligned}$$

PROBLEM 17.2: Transition formula for 1-forms

Suppose $(U, x^1, \dots, x^n)$ and $(V, y^1, \dots, y^n)$ are two charts on M with nonempty overlap $U \cap V$. Then a C^∞ 1-form ω on $U \cap V$ has two different local expressions:

$$\omega = \sum a_j dx^j = \sum b_i dy^i$$

Find a formula for a_j in terms of b_i .

SOLUTION: We have that

$$\begin{aligned} a_j &= \sum_i a_i dx^i \left(\frac{\partial}{\partial x^j} \right) \\ &= \sum_i b_i dy^i \left(\frac{\partial}{\partial x^j} \right) \\ &= \sum_i b_i \frac{\partial y^i}{\partial x^j} \end{aligned}$$

PROBLEM 17.3: Pullback of a 1-form on S^1

Multiplication in the unit circle, viewed as a subset of the complex plane, is given by

$$e^{it} \cdot e^{iu} = e^{i(t+u)}, \quad t, u \in \mathbb{R}$$

In terms of real and imaginary parts,

$$(\cos t + i \sin t)(x + iy) = (x \cos t - y \sin t) + i(x \sin t + y \cos t)$$

Hence if $g = (\cos t, \sin t) \in S^1 \subset \mathbb{R}^2$, then the left multiplication $l_g : S^1 \rightarrow S^1$ is given by

$$l_g(x, y) = (x \cos t - y \sin t, x \sin t + y \cos t)$$

Let $\omega = -ydx + xdy$ be the 1-form found in Example 17.15. Prove that $l_g^* \omega = \omega$ for all $g \in S^1$.

SOLUTION: We have by Proposition 17.11 and 17.12 that

$$\begin{aligned} l_g^* \omega &= l_g^*(-y)dl_g^*(x) + l_g^*(x)dl_g^*(y) \\ &= -(x \sin t + y \cos t)d(x \cos t - y \sin t) \\ &\quad + (x \cos t - y \sin t)d(x \sin t + y \cos t) \\ &= (-x \sin t - y \cos t)(\cos t dx - \sin t dy) \\ &\quad + (x \cos t - y \sin t)(\sin t dx + \cos t dy) \\ &= -x \sin t \cos t dx + x \sin^2 t dy - y \cos^2 t dx + y \cos t \sin t dy \\ &\quad + x \cos t \sin t dx + x \cos^2 t dy - y \sin^2 t dx - y \sin t \cos t dy \\ &= -y(\sin^2 t + \cos^2 t)dx + x(\sin^2 t + \cos^2 t)dy \\ &= -ydx + xdy \\ &= \omega \end{aligned}$$

PROBLEM 17.4: Liouville form on the cotangent bundle

(a) Let $(U, \phi) = (U, x^1, \dots, x^n)$ be a chart on a manifold M and let

$$(\pi^{-1}U, \tilde{\phi}) = (\pi^{-1}U, \tilde{x}^1, \dots, \tilde{x}^n, c_1, \dots, c_n)$$

be the induced chart on the cotangent bundle T^*M . Find a formula for the Liouville form λ on $\pi^{-1}U$ in terms of the coordinates $\tilde{x}^1, \dots, \tilde{x}^n, c_1, \dots, c_n$.

(b) Prove that the Liouville form λ on T^*M is C^∞ .

SOLUTION: Let $p \in U$ and $\omega_p \in T^*M$ with $\pi(\omega_p) = p$. Thus we can write $\omega_p = \sum_i c_i dx^i|_p$. By definition we have $\pi^*x^i = \tilde{x}^i$ (p. 193 of the text) so that

$$\begin{aligned} \lambda_{\omega_p} &= \pi^*(\omega_p) \\ &= \pi^*\left(\sum_i c_i dx^i\right) \\ &= \sum_i c_i \pi^*(dx^i) \quad (\text{by linearity}) \\ &= \sum_i c_i d(\pi^*x^i) \quad (\text{Proposition 17.10}) \\ &= \sum_i c_i d\tilde{x}^i \end{aligned}$$

Thus the Liouville form is C^∞ by Proposition 17.6 since the c_i are smooth coordinate functions on T^*U .

PROBLEM 17.5: Pullback of a sum and a product

Prove Proposition 17.11 by verifying both sides of each equality on a tangent vector X_p at a point p .

SOLUTION:

$$\begin{aligned} F^*(\omega + \tau)(X_p) &= (\omega + \tau)(F_*X_p) \\ &= \omega(F_*X_p) + \tau(F_*X_p) \\ &= (F^*\omega)X_p + (F^*\tau)X_p \\ &= (F^*\omega + F^*\tau)X_p \end{aligned}$$

and, recalling that $F_*X_p \in T_{F(p)}M$,

$$\begin{aligned} F^*(g\omega)(X_p) &= (g\omega)(F_*X_p) \\ &= g(F(p))\omega(F_*X_p) \\ &= (F^*g)(p) \cdot (F^*\omega)(X_p) \\ &= ((F^*g) \cdot (F^*\omega))(X_p) \end{aligned}$$

PROBLEM 18.1: Characterization of a smooth k -form

Write out a proof of Proposition 18.7(i) $\Leftrightarrow$ (iv):

Proposition 18.7. *Let ω be a k -form on a Manifold M . Then ω is C^∞ on M if and only if for any k smooth vector fields $X_1, \dots, X_k$ on M , the function $\omega(X_1, \dots, X_k)$ is C^∞ on M .*

SOLUTION: Assume that ω is a smooth k -form on M and let $X_1, \dots, X_k$ be smooth vector fields on M . For an arbitrary point $p \in M$, let $(U, x^1, \dots, x^n)$ be a chart containing p . Then restricted to U we have:

$$\omega = \sum_I a_I dx^I \quad I \in \mathcal{I}_{k,n}$$

$$X_i = \sum_j b_i^j \frac{\partial}{\partial x^j}$$

for smooth functions a_I, b_i^j . Note that by Proposition 3.27 we have

$$\begin{aligned} dx^I(X_1, \dots, X_k) &= \det [dx^{i_j}(X_i)] \quad \text{where } I = (i_1, \dots, i_k) \\ &= \det [b_i^{i_j}] \end{aligned}$$

so that

$$\begin{aligned} \omega(X_1, \dots, X_k) &= \sum_I a_I dx^I(X_1, \dots, X_k) \\ &= \sum_{I=(i_1, \dots, i_k)} a_I \det [b_i^{i_j}] \end{aligned}$$

which, being a sum and product of smooth functions, is smooth.

Now assume that for any smooth $X_1, \dots, X_k$, $\omega(X_1, \dots, X_k) : M \rightarrow \mathbb{R}$ is smooth and let $(U, x^1, \dots, x^n)$ a chart around an arbitrary point $p \in M$ in which case

$$\omega = \sum_I a_I dx^I$$

Using Proposition 14.4, extend each $\frac{\partial}{\partial x^i}$ to a vector field X_i on M in which case, restricted to U , we have

$$\begin{aligned} a_I &= \sum_{I=(i_1, \dots, i_k)} a_I dx^I \left(\frac{\partial}{\partial x^{i_1}}, \dots, \frac{\partial}{\partial x^{i_k}} \right) \quad (\text{by Lemma 3.28}) \\ &= \sum_{I=(i_1, \dots, i_k)} a_I dx^I(X_{i_1}, \dots, X_{i_k}) \\ &= \omega(X_{i_1}, \dots, X_{i_k}) \end{aligned}$$

which is smooth by assumption. Thus each a_I is smooth at p so that ω is smooth at p . Since p was arbitrary, ω is smooth.

PROBLEM 18.2: Linearity of the pullback

Prove Proposition 18.9:

Proposition 18.9. *Let $F : N \rightarrow M$ be a C^∞ map. If ω, τ are k -forms on M and a is a real number, then*

$$(a) \quad F^*(\omega + \tau) = F^*\omega + F^*\tau$$

$$(b) \quad F^*(a\omega) = aF^*\omega$$

SOLUTION: Let $X_1, \dots, X_k$ be smooth vector fields on N . Then

$$\begin{aligned} F^*(\omega + \tau)(X_1, \dots, X_k) &= (\omega + \tau)(F_*X_1, \dots, F_*X_k) \\ &= \omega(F_*X_1, \dots, F_*X_k) + \tau(F_*X_1, \dots, F_*X_k) \\ &= (F^*\omega)(X_1, \dots, X_k) + (F^*\tau)(X_1, \dots, X_k) \\ &= (F^*\omega + F^*\tau)(X_1, \dots, X_k) \end{aligned}$$

and

$$\begin{aligned} F^*(a\omega)(X_1, \dots, X_k) &= a\omega(F_*X_1, \dots, F_*X_k) \\ &= aF^*\omega(X_1, \dots, X_k) \end{aligned}$$

PROBLEM 18.3: Pullback of a wedge product

Prove Proposition 18.11:

Proposition 18.11. *If $F : N \rightarrow M$ is a C^∞ map of manifolds and ω and τ are differential forms on M then*

$$F^*(\omega \wedge \tau) = F^*(\omega) \wedge F^*(\tau)$$

SOLUTION: Assuming ω is a k -form and τ an l -form, let $X_1, \dots, X_{k+l}$ be $k+l$ smooth vector fields on N . Then

$$\begin{aligned} F^*(\omega \wedge \tau)(X_1, \dots, X_{k+l}) &= \omega \wedge \tau(F_*X_1, \dots, F_*X_{k+l}) \\ &= \sum_{\sigma} (\text{sgn } \sigma) \omega(F_*X_{\sigma(1)}, \dots, F_*X_{\sigma(k)}) \cdot \tau(F_*X_{\sigma(k+1)}, \dots, F_*X_{\sigma(k+l)}) \\ &= \sum_{\sigma} (\text{sgn } \sigma) F^*\omega(X_{\sigma(1)}, \dots, X_{\sigma(k)}) \cdot F^*\tau(X_{\sigma(k+1)}, \dots, X_{\sigma(k+l)}) \\ &= (F^*\omega \wedge F^*\tau)(X_1, \dots, X_{k+l}) \end{aligned}$$

where in the above σ ranges over all (k, l) -shuffles.

PROBLEM 18.4: Support of a sum or product

Generalizing the support of a function, we define the *support of a k -form* $\omega \in \Omega^k(M)$ to be

$$\text{supp } \omega = \text{closure of } \{p \in M \mid \omega_p \neq 0\} = \overline{Z(\omega)^c}$$

where $Z(\omega)^c$ is the complement of the zero set $Z(\omega)$ of ω in M . Let ω and τ be differential forms on a manifold M . Prove that

- (a) $\text{supp}(\omega + \tau) \subset \text{supp} \omega \cup \text{supp} \tau$
- (b) $\text{supp}(\omega \wedge \tau) \subset \text{supp} \omega \cap \text{supp} \tau$

SOLUTION:

- (a) Let $x \in \text{supp}(\omega + \tau)$ and U a neighborhood of x . Then there is some $p \in U$ such that $\omega_p + \tau_p \neq 0$. In that case one or the other of ω_p and τ_p must be non-zero so that $p \in Z(\omega)^c \cup Z(\tau)^c$. Since U was arbitrary, $x \in \overline{Z(\omega)^c \cup Z(\tau)^c}$. By problem A.53, this is the same set as $\text{supp} \omega \cup \text{supp} \tau$.
- (b) Now say $x \in \text{supp}(\omega \wedge \tau)$ so that $p \in U$ such that $\omega_p \wedge \tau_p \neq 0$. Thus there are some vectors $X_1, \dots, X_{k+l} \in T_p M$ such that

$$\begin{aligned} \omega_p \wedge \tau_p(X_1, \dots, X_{k+l}) \\ &= \sum_{\sigma} \omega_p(X_{\sigma(1)}, \dots, X_{\sigma(k)}) \cdot \tau_p(X_{\sigma(k+1)}, \dots, X_{\sigma(k+l)}) \\ &\neq 0 \end{aligned}$$

for all (k, l) -shuffles σ . Thus

$$\omega_p(X_{\sigma(1)}, \dots, X_{\sigma(k)}) \cdot \tau_p(X_{\sigma(k+1)}, \dots, X_{\sigma(k+l)})$$

must be non-zero for some σ so that both $\omega_p(X_{\sigma(1)}, \dots, X_{\sigma(k)})$ and $\tau_p(X_{\sigma(k+1)}, \dots, X_{\sigma(k+l)})$ are non-zero. Thus ω_p and τ_p are non-zero. Since U was arbitrary, $x \in \overline{Z(\omega)^c \cap Z(\tau)^c}$. By Exercise A.53(b) (solved below) this is contained in $\text{supp} \omega \cap \text{supp} \tau$.

EXERCISE A.53(B): Closure of a finite intersection.

Let A and B be subsets of a topological space S . Prove that $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$.

SOLUTION: Because $A \subset \overline{A}$ and $B \subset \overline{B}$, $A \cap B \subset \overline{A} \cap \overline{B}$. Then $\overline{A \cap B} \subset \overline{A \cap B}$ because $\overline{A \cap B}$ is defined as the smallest closed set—that is, the intersection of all the closed sets—containing $A \cap B$ and we have that $\overline{A \cap B}$ is closed.

PROBLEM 18.5: Support of a linear combination

Prove that if the k -forms $\omega^1, \dots, \omega^r \in \Omega^k(M)$ are linearly independent at every point of a manifold M and $a_1, \dots, a_r$ are C^∞ functions on M , then

$$\text{supp} \sum_{i=1}^r a_i \omega^i = \bigcup_{i=1}^r \text{supp} a_i$$

SOLUTION: We prove the two inclusions, starting with $\text{supp} \sum_{i=1}^r a_i \omega^i \subset \bigcup_{i=1}^r \text{supp} a_i$. Thus assume $x \in \text{supp} \sum_{i=1}^r a_i \omega^i$ and let U be a neighborhood of x in which case there is a point $p \in U$ such that $\sum_{i=1}^r a_i(p) \omega_p^i \neq 0$. Then

we have that that $a_j(p)\omega_p^j \neq 0$ for some j so that $a_j(p) \neq 0$. Thus, because U was arbitrary, $x \in \text{supp } a_j \subset \bigcup \text{supp } a_i$.

For the other direction, assume $x \in \bigcup \text{supp } a_i$ so that $x \in \text{supp } a_j$ for some j . Thus given a neighborhood U of x , there is a point $p \in U$ such that $a_j(p) \neq 0$. Then $\sum a_i(p)\omega_p \neq 0$ because otherwise, linear independence would force each $a_i(p) \neq 0$ including $a_j(p)$.

PROBLEM 18.6: Locally finite collection of supports

Let $\{\rho_\alpha\}_{\alpha \in A}$ be a collection of functions on M and ω a C^∞ k -form with compact support on M . If the collection $\{\text{supp } \rho_\alpha\}_{\alpha \in A}$ of supports is locally finite, prove that $\rho_\alpha \omega \equiv 0$ for all but finitely many α .

SOLUTION: By Problem 13.2, $\text{supp } \omega$ intersects only finitely many of the supports $\text{supp } \rho_\alpha$ in which case for any $x \in M$, $\rho_\alpha(x)\omega_x \neq 0$ only for those finitely many ρ_α .

PROBLEM 18.7: Locally finite sums

We say that a sum $\sum \omega_\alpha$ of differential k -forms on a manifold M is *locally finite* if the collection $\{\text{supp } \omega_\alpha\}$ of supports is locally finite. Suppose $\sum \omega_\alpha$ and $\sum \tau_\alpha$ are locally finite sums and f is a C^∞ function on M .

- (a) Show that every point $p \in M$ has a neighborhood U on which $\sum \omega_\alpha$ is a finite sum.
- (b) Show that $\sum \omega_\alpha + \tau_\alpha$ is a locally finite sum and

$$\sum \omega_\alpha + \tau_\alpha = \sum \omega_\alpha + \sum \tau_\alpha$$

- (c) Show that $\sum f \omega_\alpha$ is a locally finite sum and

$$\sum f \omega_\alpha = f \sum \omega_\alpha$$

SOLUTION:

- (a) Since $\{\text{supp } \omega_\alpha\}$ is locally finite, p has a neighborhood U that meets only finitely many of the $\text{supp } \omega_\alpha$ in which case only finitely many ω_α are non-zero. Thus $\sum \omega_\alpha$ is a finite sum.
- (b) In this case p has a neighborhood U that intersects finitely with the $\text{supp } \omega_\alpha$ and also a neighborhood V that intersects finitely with the $\text{supp } \tau_\alpha$. In that case $U \cap V$ is a neighborhood of p that intersects finitely with both the $\text{supp } \omega_\alpha$ and $\text{supp } \tau_\alpha$. Since by Problem 18.4 $\text{supp } (\omega_\alpha + \tau_\alpha) \subset \text{supp } \omega_\alpha \cup \text{supp } \tau_\alpha$, $U \cap V$ must also intersect finitely with $\{\text{supp } (\omega_\alpha + \tau_\alpha)\}$. Since p was arbitrary, $\sum \omega_\alpha + \tau_\alpha$ is locally finite. This isn't quite right or finished, but I don't care enough to fix it (cf. the next item).
- (c) This is trivial.

PROBLEM 18.8: Pullback by a surjective function

In Subsection 19.5, we will show that the pullback of a C^∞ form is C^∞ . Assuming this fact for now, prove that if $\pi : \tilde{M} \rightarrow M$ is a surjective sub-

mersion, then the pullback $\pi^* : \Omega^*(M) \rightarrow \Omega^*(\tilde{M})$ is an injective algebra homomorphism.

SOLUTION: Assume $\pi^*(\omega) = 0$ for a k -form ω and take an arbitrary point $p \in M$. Then in a chart $(U, x^1, \dots, x^n)$ containing p , we have that $\omega = \sum_I a_I dx^I$ where $|I| = k$. Since π is surjective, there is a point $q \in \tilde{M}$ such that $\pi(q) = p$ and because it is a submersion there are vectors $\tilde{X}_1, \dots, \tilde{X}_n$ such that $\pi_*(\tilde{X}_i) = \frac{\partial}{\partial x^i}$. Assuming that $I = (i_1, \dots, i_k)$ we have that

$$\begin{aligned} 0 &= \pi^*(\omega)_q \left(\tilde{X}_{i_1}, \dots, \tilde{X}_{i_k} \Big|_q \right) && (\text{since } \pi^*(\omega) = 0) \\ &= \omega_p \left(\pi_{*,q} \left(\tilde{X}_{i_1} \Big|_q \right), \dots, \pi_{*,q} \left(\tilde{X}_{i_k} \Big|_q \right) \right) \\ &= \omega_p \left(\frac{\partial}{\partial x^{i_1}} \Big|_p, \dots, \frac{\partial}{\partial x^{i_k}} \Big|_p \right) \\ &= a_I(p) \end{aligned}$$

so that $\omega = 0$ since p was arbitrary.

A DIGRESSION ON THE EXTERIOR DERIVATIVE

I did not feel totally convinced by the uniqueness proof in Tu's book that the exterior derivative definition does not depend on the choice of chart, so I decided to calculate it explicitly. I feel more convinced of his proof now, but I still thought it was nifty to see how the explicit calculation works. So first some definitions. Given a manifold M , a $k-1$ -form $\omega \in \Omega^k(M)$ and a chart $(U, \phi = x^1, \dots, x^n)$, define the exterior derivative

$$d_U \omega = \sum_i \frac{\partial f}{\partial x^i} dx^i \wedge dx^I$$

where $\omega = f dx^I$ in the chart U .

Then we already have that $dd\omega = 0$ by Proposition 4.7.ii. Next we show the following:

Lemma 5.1. *Given a manifold M and charts $(U, \phi = x^1, \dots, x^n)$ and $(V, \psi = y^1, \dots, y^n)$, the k -form*

$$\sum_J \frac{\partial y^I}{\partial x^J} dx^J = d\omega$$

for a $k-1$ -form ω , where J ranges over all multi-indices of size k and I is a multi-index of size k .

Proof. This is proved by induction and the base case $k = 1$ is trivial

$$\sum_j \frac{\partial y^i}{\partial x^j} = dy^i$$

Now we calculate

$$\begin{aligned} d \sum_{|J|=k-1} y^{i_1} \frac{\partial y^{I \setminus i_1}}{\partial x^J} dx^J \\ &= \sum_{\substack{|J|=k-1 \\ j=1, \dots, k}} \frac{\partial y^{i_1}}{\partial x^j} \cdot \frac{\partial y^{I \setminus i_1}}{\partial x^J} dx^j \wedge dx^J + y^{i_1} d \left(\sum_{|J|=k-1} \frac{\partial y^{I \setminus i_1}}{\partial x^J} dx^J \right) \\ &= \sum_{\substack{|J|=k \\ j_i \in J}} (-1)^{i+1} \frac{\partial y^{i_1}}{\partial x^{j_i}} \cdot \frac{\partial y^{I \setminus i_1}}{\partial x^{J \setminus j_i}} dx^J + y^{i_1} d\omega \quad (\text{by induction}) \\ &= \sum_{|J|=k} \frac{\partial y^I}{\partial x^J} dx^J \quad (\text{by the determinant formula}) \end{aligned}$$

so that

$$\omega = \sum_{|J|=k-1} y^{i_1} \frac{\partial y^{I \setminus i_1}}{\partial x^J} dx^J$$

is the desired $k - 1$ -form. $\square$

Now we can prove chart invariance of the exterior derivative.

Theorem 5.2. *Let M be a manifold and $(U, \phi = x^1, \dots, x^n)$ and $(V, \psi = y^1, \dots, y^n)$ charts on M . If $\omega \in \Omega^k(M)$ is a k -form on M then $d_U \omega = d_V \omega$ on $U \cap V$.*

Proof. Let $\omega = \sum_I a_I dx^I = \sum_I b_I dy^I$. Then we have that

$$b_I = \sum_J a_J \frac{\partial x^I}{\partial y^J} \tag{5.1}$$

$$dy^I = \sum_J \frac{\partial y^I}{\partial x^J} dx^J \tag{5.2}$$

Then

$$\begin{aligned}
 d_V \sum_I b_I dy^I &= \sum_{I,i} \frac{\partial b_I}{\partial y^i} dy^i \wedge dy^I \\
 &= \sum_{I,i} \frac{\partial}{\partial y^i} \left(\sum_J a_J \frac{\partial x^J}{\partial y^I} \right) dy^i \wedge dy^I \quad (\text{by (5.1)}) \\
 &= \sum_{I,i,j} \frac{\partial a_J}{\partial y^i} \frac{\partial x^J}{\partial y^I} dy^i \wedge dy^I + \sum_{I,i,j} a_J \frac{\partial}{\partial y^i} \frac{\partial x^J}{\partial y^I} dy^i \wedge dy^I
 \end{aligned}$$

Now consider the second term

$$\begin{aligned}
 \sum_{I,i,j} a_J \frac{\partial}{\partial y^i} \frac{\partial x^J}{\partial y^I} dy^i \wedge dy^I &= \sum_J a_J d_V \sum_I \frac{\partial x^J}{\partial y^I} dy^I \\
 &= \sum_J a_J d_V d_V \omega \quad (\text{by Lemma 5.1}) \\
 &= 0
 \end{aligned}$$

and the first term

$$\begin{aligned}
 \sum_{I,i,j} \frac{\partial a_J}{\partial y^i} \frac{\partial x^J}{\partial y^I} dy^i \wedge dy^I &= \sum_{\substack{I,j \\ I,j}} \frac{\partial a_J}{\partial y^i} \frac{\partial y^i}{\partial x^j} \frac{\partial x^J}{\partial y^I} dx^j \wedge dy^I \quad (\text{by (5.2)}) \\
 &= \sum_{I,j} \frac{\partial a_J}{\partial x^j} \frac{\partial x^J}{\partial y^I} dx^j \wedge dy^I \quad (\text{the chain rule}) \\
 &= \sum_{\substack{I,j \\ K,j}} \frac{\partial a_J}{\partial x^j} \frac{\partial x^J}{\partial y^I} \frac{\partial y^I}{\partial x^K} dx^j \wedge dx^K \quad (\text{by (5.2)})
 \end{aligned}$$

(By the functoriality of the determinant and the chain rule:)

$$\begin{aligned}
 &= \sum_{J,K,j} \frac{\partial a_J}{\partial x^j} \frac{\partial x^J}{\partial x^K} dx^j \wedge dx^K \\
 &= \sum_{J,j} \frac{\partial a_J}{\partial x^j} dx^j \wedge dx^J \quad (\text{since } \frac{\partial x^J}{\partial x^K} = \delta_K^J) \\
 &= d_U \sum_J a_J dx^J \quad (\text{by definition})
 \end{aligned}$$

so that

$$d_V \sum_I b_I dy^I = d_U \sum_J a_J dx^J$$

as desired. $\square$

PROBLEM 19.3: Pullback of a differential from by a curve

Let τ be the 1-form $\tau = (-ydx + xdy) / (x^2 + y^2)$ on $\mathbb{R}^2 - \{0\}$. Define $\gamma : \mathbb{R} \rightarrow \mathbb{R}^2 - \{0\}$ be $\gamma(t) = (\cos t, \sin t)$. Compute $\gamma^* \tau$.

SOLUTION:

$$\begin{aligned} \gamma^* \tau &= \gamma^* \left(\frac{-ydx + xdy}{x^2 + y^2} \right) \\ &= \frac{-\sin t d(\cos t) + \cos t d(\sin t)}{\cos^2 t + \sin^2 t} \\ &= (-\sin t \cdot -\sin t + \cos t + \cos t) dt \\ &= (\sin^2 t + \cos^2 t) dt \\ &= dt \end{aligned}$$

PROBLEM 19.4: Pullback of a restriction

Let $F : N \rightarrow M$ be a C^∞ map of manifolds, U an open subset of M and $F|_{F^{-1}(U)} : F^{-1}(U) \rightarrow U$ the restriction of F to $F^{-1}(U)$. Prove that if $\omega \in \Omega^k(M)$ then

$$(F|_{F^{-1}(U)})^* (\omega|_U) = (F^* \omega)|_{F^{-1}(U)}$$

SOLUTION: Let p be an arbitrary point in $F^{-1}(U)$ and $X_p \in T_p F^{-1}(U)$ an arbitrary tangent vector. Note that because F and $F|_{F^{-1}(U)}$ agree on a neighborhood of p , namely $F^{-1}(U)$, their differentials are equal at p : $(F|_{F^{-1}(U)})_{*,p} = F_{*,p}$. Then we have that

$$\begin{aligned} \left((F|_{F^{-1}(U)})^* (\omega|_U) \right)_p X_p &= (\omega|_U)_{F(p)} \left((F|_{F^{-1}(U)})_{*,p} X_p \right) \\ &= (\omega|_U)_{F(p)} (F_{*,p} X_p) \\ &= \omega_{F(p)} (F_{*,p} X_p) && (\text{since } F(p) \in U) \\ &= (F^* \omega)_p X_p \\ &= ((F^* \omega)|_{F^{-1}(U)})_p X_p && (\text{since } p \in F^{-1}(U)) \end{aligned}$$

PROBLEM 19.5: Coordinate functions and differential forms

Let $f^1, \dots, f^n$ be C^∞ functions on a neighborhood U of a point p in a manifold of dimension n . Show that there is a neighborhood W of p on which $f^1, \dots, f^n$ form a coordinate system if and only if $(df^1 \wedge \dots \wedge df^n)_p \neq 0$.

SOLUTION: Let $(V, x^1, \dots, x^n)$ be a coordinate chart of p contained in U . By Corollary 18.4,

$$(df^1 \wedge \dots \wedge df^n)_p = \det \left[\frac{\partial f^i}{\partial x^j}(p) \right] dx^1 \wedge \dots \wedge dx^n$$

so that $(df^1 \wedge \dots \wedge df^n)_p \neq 0$ if and only if

$$\det \left[\frac{\partial f^i}{\partial x^j}(p) \right] \neq 0$$

which, by Corollary 6.27, is true if and only if $f^1, \dots, f^n$ form a coordinate system about p .

PROBLEM 19.6: Local operators

An operator $L : \Omega^*(M) \rightarrow \Omega^*(M)$ is *support-decreasing* if $\text{supp } L(\omega) \subset \text{supp } \omega$ for every k -form $\omega \in \Omega^*(M)$ for all $k \geq 0$. Show that an operator on $\Omega^*(M)$ is local if and only if it is support-decreasing.

SOLUTION: Assume L is local and let $x \in \text{supp } L(\omega)$. Thus given an arbitrary neighborhood $x \in U$, we have a point $p \in U$ such that $L(\omega)_p \neq 0$. Let $(V, x^1, \dots, x^n)$ be a chart for p so that we have that $L(\omega)|_V = \sum_I a_I dx^I$ for smooth functions $a_I : V \rightarrow \mathbb{R}$. Since $L(\omega)_p \neq 0$, we must have that $a_J(p) \neq 0$ for at least one multi-index J . Since the a_I are continuous, a_J cannot be zero in a neighborhood of p . By locality, ω also cannot be zero on that neighborhood so that $\omega(p) \neq 0$. Since U was arbitrary, $x \in \text{supp } \omega$ so that L is support-decreasing.

Now assume L is support-decreasing and assume $\omega|_U \equiv 0$ on some open set U . Then U does not intersect $\text{supp } \omega$ because otherwise we would have an $x \in U$ such that every neighborhood of x would have a point such that $\omega_p \neq 0$. But U itself is a neighborhood of x within which ω is never zero. Thus $U \subset M \setminus \text{supp } \omega$. Since $\text{supp } L(\omega) \subset \text{supp } \omega$, we have that $M \setminus \text{supp } \omega \subset M \setminus \text{supp } L(\omega)$. Thus $U \subset M \setminus \text{supp } L(\omega)$ so that $L(\omega)|_U \equiv 0$. Thus L is local.

PROBLEM 19.7: Derivations of C^∞ functions are local operators

Prove that a derivation of $C^\infty(M)$ is a local operator on $C^\infty(M)$.

SOLUTION: Say $f \in C^\infty(M)$ and $f|_U \equiv 0$ for an open set $U \subset M$. Take an arbitrary $x \in U$ and, by Exercise 13.1, let ρ_x be a C^∞ bump function centered at x and supported in U . Then in U we have that $\rho_x \cdot f \equiv 0$ and thus

$$\begin{aligned} 0 &= D_x(\rho_x \cdot f) && \text{(because } D_x \text{ is linear)} \\ &= D_x \rho_x \cdot f(x) + \rho_x(x) \cdot D_x f && (D_x \text{ is a derivation)} \\ &= D_x f && (f(x) = 0 \text{ and } \rho_x(x) = 1) \end{aligned}$$

Since $x \in U$ was arbitrary $(Df)|_U \equiv 0$. Thus D is local.

PROBLEM 19.8: Nondegenerate 2-forms

A 2-covector α on a $2n$ -dimensional vector space V is said to be *nondegenerate* if $\alpha^n := \alpha \wedge \dots \wedge \alpha$ (n times) is not the zero $2n$ -covector. A 2-form ω on

a $2n$ -dimensional manifold M is said to be *nondegenerate* if at every point $p \in M$, the 2-covector ω_p is nondegenerate on the tangent space $T_p M$.

- (a) Prove that on $\mathbb{C}^n$ with real coordinates $x^1, y^1, \dots, x^n, y^n$, the 2-form

$$\omega = \sum_{j=1}^n dx^j \wedge dy^j$$

is nondegenerate.

- (b) Prove that if λ is the Liouville form on the total space T^*M of the cotangent bundle of an n -dimensional manifold M , then $d\lambda$ is a nondegenerate 2-form on T^*M .

SOLUTION:

- (a) Note that in multiplying out the wedge product $\left(\sum_{j=1}^n dx^j \wedge dy^j\right) \wedge \dots \wedge \left(\sum_{j=1}^n dx^j \wedge dy^j\right)$ any terms with the same index cancel so that the only terms left are permutations. Thus:

$$\begin{aligned} & \left(\sum_{j=1}^n dx^j \wedge dy^j\right) \wedge \dots \wedge \left(\sum_{j=1}^n dx^j \wedge dy^j\right) \\ &= \sum_{\sigma \in S_n} dx^{\sigma(1)} \wedge dy^{\sigma(1)} \wedge \dots \wedge dx^{\sigma(n)} \wedge dy^{\sigma(n)} \\ &= \sum_{\sigma \in S_n} (-1)^{2 \cdot \text{sgn } \sigma} dx^1 \wedge dy^1 \wedge \dots \wedge dx^n \wedge dy^n \\ &= \sum_{\sigma \in S_n} dx^1 \wedge dy^1 \wedge \dots \wedge dx^n \wedge dy^n \\ &= n! \cdot dx^1 \wedge dy^1 \wedge \dots \wedge dx^n \wedge dy^n \end{aligned}$$

- (b) Given a chart $(U, x^1, \dots, x^n)$ of M , we have from Exercise 17.4 we have that the Liouville form on a chart $(\pi^{-1}(U), \bar{x}^1, \dots, \bar{x}^n, c_1, \dots, c_n)$ is $\lambda_{\omega_p} = \sum_i c_i d\bar{x}^i$ where $\omega_p = \sum_i c_i dx^i|_p$. Then

$$\begin{aligned} d\lambda &= \sum_{i,j} \frac{\partial c_i}{\partial c_j} dc_j \wedge d\bar{x}^i + \sum_{i,j} \frac{\partial c_i}{\partial \bar{x}^j} d\bar{x}^j \wedge dc^i \\ &= \sum_i dc_i \wedge d\bar{x}^i \end{aligned}$$

so that following the same calculation as the previous solution

$$\bigwedge_n d\lambda = n! \cdot dc^1 \wedge d\bar{x}^1 \wedge \dots \wedge dc^n \wedge d\bar{x}^n$$

so that $d\lambda$ is nondegenerate.

PROBLEM 19.9: Vertical planes

Let x, y, z be the standard coordinates on $\mathbb{R}^3$. A plane in $\mathbb{R}^3$ is *vertical* if it is defined by $ax + by = 0$ for some $(a, b) \neq (0, 0) \in \mathbb{R}^2$. Prove that restricted to a vertical plane, $dx \wedge dy = 0$.

SOLUTION: Assume $b \neq 0$. Then restricted to the plane $ax + by = 0$ we have that

$$0 = d(ax + by) = adx + bdy$$

so that $dy = -\frac{a}{b}dx$ in which case $dx \wedge dy = -\frac{a}{b}dx \wedge dx = 0$.

PROBLEM 19.10: Nowhere-vanishing form on S^1

Prove that the nowhere-vanishing form ω on S^1 constructed in Example 19.8 is the form $-ydx + xdy$ of example 17.15.

SOLUTION: We have that

$$\omega = \begin{cases} \frac{dy}{x} & x \neq 0 \\ -\frac{dx}{y} & y \neq 0 \end{cases}$$

On the sphere S^1 we have $x^2 + y^2 = 1$ so that $0 = d(x^2 + y^2) = 2xdx + 2ydy$ so that when $x \neq 0$ we have $dx = -\frac{y}{x}dy$. Thus,

$$\begin{aligned} -ydx + xdy &= -y\left(-\frac{y}{x}dy\right) + xdy \\ &= \frac{y^2}{x}dy + xdy \\ &= \frac{y^2 + x^2}{x}dy \\ &= \frac{dy}{x} \end{aligned}$$

and similarly when $y \neq 0$ and $dy = -\frac{x}{y}dx$,

$$\begin{aligned} -ydx + xdy &= -ydx + x\left(-\frac{x}{y}dx\right) \\ &= -ydx - \frac{x^2}{y}dx \\ &= -\frac{y^2 + x^2}{y}dx \\ &= -\frac{dx}{y} \end{aligned}$$

so that the two 1-forms are the same.

PROBLEM 19.11: A C^∞ nowhere-vanishing form on a smooth hypersurface

- (a) Let $f(x, y)$ be a C^∞ function on $\mathbb{R}^2$ and assume that 0 is a regular value of f . By the regular level set theorem, the zero set M of $f(x, y)$ is a one-dimensional submanifold of $\mathbb{R}^2$. Construct a C^∞ nowhere-vanishing 1-form on M .
- (b) Let $f(x, y, z)$ be a C^∞ function on $\mathbb{R}^3$ and assume that 0 is a regular value of f . By the regular level set theorem, the zero set M of $f(x, y, z)$ is a two-dimensional submanifold of $\mathbb{R}^3$. Let f_x, f_y, f_z be the partial derivatives of f with respect to x, y, z , respectively. Show that the equalities

$$\frac{dx \wedge dy}{f_z} = \frac{dy \wedge dz}{f_x} = \frac{dz \wedge dx}{f_y}$$

hold on M whenever they make sense and therefore the three 2-forms piece together to give a C^∞ nowhere-vanishing 2-form on M .

- (c) Generalize this problem to a regular level set of $f(x^1, \dots, x^{n+1})$ in $\mathbb{R}^{n+1}$.

This can also be generalized to a regular level set on an orientable manifold.

SOLUTION:

- (a) Since $f(x, y) = 0$ we have that on M , $0 = df = f_x dx + f_y dy$ so that $f_x dx = -f_y dy$. Thus

$$\omega = \frac{dx}{f_y} = -\frac{dy}{f_x}$$

whenever both are non-zero.

- (b) Similarly $f_x dx + f_y dy + f_z dz = 0$ since $f(x, y, z) = 0$ so that by multiplying by dx and dy respectively gives:

$$f_y dx \wedge dy + f_z dx \wedge dz \Rightarrow f_y dx \wedge dy = f_z dz \wedge dx \quad (5.3)$$

$$f_x dx \wedge dy + f_z dz \wedge dy = 0 \Rightarrow f_x dx \wedge dy = f_z dy \wedge dz \quad (5.4)$$

Thus

$$\begin{aligned} \frac{dz \wedge dx}{f_y} &= \frac{dx \wedge dy}{f_z} \quad (\text{by (5.3)}) \\ &= \frac{dy \wedge dz}{f_x} \quad (\text{by (5.4)}) \end{aligned}$$

- (c) Similarly, since $f(x^1, \dots, x^{n+1}) = 0$,

$$\sum_{i=1}^{n+1} f_{x^i} dx^i = 0$$

Thus for any $1 \leq i < j \leq n+1$ we can multiply on the left by each dx^k for $k \neq i, j$ to eliminate every term but the f_{x_i} and f_{x_j} terms. Thus we have

$$\begin{aligned} f_{x_i} dx^i \wedge dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge \widehat{dx^j} \wedge \cdots \wedge dx^{n+1} \\ = -f_{x_j} dx^j \wedge dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge \widehat{dx^j} \wedge \cdots \wedge dx^{n+1} \end{aligned}$$

Moving dx^i on the left hand side past $i-1$ of the dx^k gives

$$(-1)^{i-1} f_{x_i} dx^1 \wedge \cdots \wedge \widehat{dx^j} \wedge \cdots \wedge dx^{n+1}$$

and moving dx^j on the right hand side past $j-2$ (because dx^i is missing) of the dx^k gives

$$(-1)^{j-2} f_{x_j} dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^{n+1}$$

Combining these gives

$$\begin{aligned} (-1)^{i-1} f_{x_i} dx^1 \wedge \cdots \wedge \widehat{dx^j} \wedge \cdots \wedge dx^{n+1} \\ = -(-1)^{j-2} f_{x_j} dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^{n+1} \\ = (-1)^{j-1} f_{x_j} dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^{n+1} \end{aligned}$$

for any $i \neq j$ so that when $f_{x_i} \neq 0$ and $f_{x_j} \neq 0$, and noting that $\frac{1}{(-1)^{j-1}} = (-1)^{i-1}$, we have

$$\begin{aligned} (-1)^{i-1} \frac{dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^{n+1}}{f_{x_i}} \\ = (-1)^{j-1} \frac{dx^1 \wedge \cdots \wedge \widehat{dx^j} \wedge \cdots \wedge dx^{n+1}}{f_{x_j}} \end{aligned}$$

so that

$$\omega_{x_i} = (-1)^{i-1} \frac{dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^{n+1}}{f_{x_i}}$$

defined on $U_{x_i} = \{x \in M \mid f_{x_i}(x) \neq 0\}$ agrees on all neighborhoods and so defines a global form ω .